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Spectrum of a multi-species asymmetric simple exclusion process on a ring

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Abstract
The spectrum of the Hamiltonian (Markov matrix) of a multi-species asymmetric simple exclusion process on a ring is studied. The dynamical exponent concerning the relaxation time is found to coincide with the one-species case. It implies that the system belongs to the Kardar–Parisi–Zhang or Edwards–Wilkinson universality classes depending on whether the hopping rate is asymmetric or symmetric, respectively. Our derivation exploits a poset structure of the particle sectors, leading to a new spectral duality and inclusion relations. The Bethe ansatz integrability is also demonstrated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years, intensive studies on non-equilibrium phenomena have been undertaken through a variety of stochastic process models of many particle systems [S1]. Typical examples are driven lattice gas systems with a simple but nonlinear interaction among constituent particles. The asymmetric simple exclusion process (ASEP) is one of the simplest driven lattice gas models proposed originally to describe the dynamics of ribosome along RNA [MGP]. In the ASEP, each site is occupied by at most one particle. Each particle is allowed to hop to its nearest-neighbor right (left) site with the rate p (q) if it is empty. The ASEP admits exact analyses of non-equilibrium properties by the matrix product ansatz and the Bethe ansatz. The matrix product form of the stationary state was first found in the open boundary case [DEHP]. Similar results have been obtained in various driven lattice gas systems in one dimension with both open and periodic boundary conditions [BE]. Applications of the Bethe ansatz [B] to the ASEP have also been successful. See for example the works [GS, K, S2, DE] for the studies of the ASEP under periodic, infinite and open boundary conditions, respectively. In the periodic boundary case, the dynamical exponent z of the relaxation time to the stationary state is $z = 3/2$ if $p \neq q$ and $z = 2$ if $p = q$ according to the analysis of the Bethe equation. This implies that the ASEP belongs to the Kardar–Parisi–Zhang (KPZ) or Edwards–Wilkinson (EW) universality classes depending on whether the hopping rate is asymmetric or symmetric, respectively.

In this paper we consider the following multi-species generalization of the ASEP on a ring of length L. A local state on each site of the ring assumes N states $1, 2, \ldots, N$. A nearest-neighbor pair of the states is interchanged with the transition rate

$$\alpha\beta \to \beta\alpha \begin{cases} p & \text{if} \quad \alpha > \beta, \\ q & \text{if} \quad \alpha < \beta. \end{cases}$$

We regard the local state $\alpha = 1$ as a vacant site and $\alpha = 2, \ldots, N$ as a site occupied by a particle of the αth kind. The dynamics is formulated in terms of the master equation $\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = H|P(t)\rangle$ on the probability vector $|P(t)\rangle$. We call the model the $(N-1)$-species ASEP or simply the multi-species ASEP. The usual ASEP corresponds to $N = 2$.

The $N^L \times N^L$ Markov matrix H will be called the 'Hamiltonian' although it is not Hermitian for $p \neq q$. (At $p = q$, it is Hermitian and coincides with the $sl(N)$ invariant Heisenberg Hamiltonian.) Its eigenvalue with the largest real part is 0 corresponding to the stationary state. The other eigenvalues contribute to the relaxation behavior through $|P(t)\rangle = \mathrm{e}^{tH}|P(0)\rangle$. Especially those with the second largest real part determine the relaxation time τ and the dynamical exponent z by the scaling behavior $\tau \sim L^z$. Our finding is that z remains the same with the usual ASEP $N = 2$. The key to the result is that the spectrum of H in any multi-species particle sector includes that for an $N = 2$ sector. We systematize such spectral inclusion relations by exploiting the poset structure of particle sectors. As a by-product we find a duality, a global aspect in the spectrum of the Hamiltonian, which is new even at the Heisenberg point $p = q$.

There are some other multi-species models with different hopping rules such as ABC model [EKKM] and AHR model [AHR, RSS], etc. The recent paper [KN] says that the AHR model still has the dynamical exponent $z = 3/2$. The multi-species ASEP in this paper is a most standard generalization of the one-species ASEP allowing the application of the Bethe ansatz. Let us comment on this point and related works to clarify the origin of integrability of the model. In [AR], a formulation by Hecke algebra was given for a wider class of stochastic models including reaction-diffusion systems. A Bethe ansatz treatment was presented in [AB]. As is well known [Ba], however, there underlies a two-dimensional integrable vertex model behind a Bethe ansatz solvable Hamiltonian. In the present model, the relevant vertex model is a special case of the Perk–Schultz model (called 'second class of models' in [PS]). It should be fair to say that the nested Bethe ansatz for the present model is originally due to Schultz [Sc]. For an account of the Perk–Schultz R-matrix in the framework of multi-parameter quantum group, see [OY] and references therein.

The layout of the paper is as follows. In section 2, we introduce the multi-species ASEP Hamiltonian together with its basic properties. In section 3, we explain how the spectral gap responsible for the relaxation time is reduced to the one-species case and determine the dynamical exponent as in (3.17). Our argument is based on spectral inclusion property (3.11) and conjecture 3.1 supported by numerical analyses. In section 4, we elucidate a new duality of the spectrum of the Hamiltonian in theorem 4.12. An intriguing feature is that it emerges only by dealing with all the basic sectors of the $N = L$-state model on the length L ring. (The term 'basic sector' will be defined in section 2.) The argument of section 4 is independent of the Bethe ansatz. In section 5, we discuss the Bethe ansatz integrability of the multi-species ASEP and the underlying vertex model including the completeness issue. We execute the nested algebraic Bethe ansatz in an arbitrary nesting order, which leads to an alternative explanation of the spectral inclusion property. Except the arbitrariness of the nesting order, section 5.1.1 is a review.

Appendix A gives a proof of the dimensional duality (theorem 4.9) needed in section 4. Appendix B is a sketch of a derivation of the stationary state by the Bethe ansatz. Appendix C contains the complete spectra of the transfer matrix and the Hamiltonian in the basic sectors with the corresponding Bethe roots for $(p, q) = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $L = 4$. They agree with the completeness conjectures in section 5.1.2.

2. Multi-species ASEP

2.1. Master equation

Consider an L-site ring $\mathbb{Z}_L$ where each site $i \in \mathbb{Z}_L$, is assigned with a variable (local state) $k_i \in \{1, \dots, N\} (N \geqslant 1)$. We introduce a stochastic model on $\mathbb{Z}_L$ such that nearest-neighbor pairs of local states $(\alpha, \beta) = (k_i, k_{i+1})$ are interchanged with the transition rate:

$$\alpha\beta \to \beta\alpha \begin{cases} p & \text{if} \quad \alpha > \beta, \\ q & \text{if} \quad \alpha < \beta, \end{cases} \tag{2.1}$$

where p and q are real non-negative parameters. More precisely, the dynamics is formulated in terms of the continuous-time master equation on the probability of finding the configuration $(k_1, \dots, k_L)$ at time t:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P(k_1, \dots, k_L; t) &= \sum_{i \in \mathbb{Z}_L} \Theta(k_{i+1} - k_i) P(k_1, \dots, k_{i-1}, k_{i+1}, k_i, k_{i+2}, \dots, k_L; t) \\ &\quad - \sum_{i \in \mathbb{Z}_L} \Theta(k_i - k_{i+1}) P(k_1, \dots, k_L; t), \end{aligned} \tag{2.2}$$

where Θ is a step function defined as

$$\Theta(x) = \begin{cases} p & (x > 0), \\ 0 & (x = 0), \\ q & (x < 0). \end{cases} \tag{2.3}$$

(Actually, $\Theta(0)$ can be set to any value.)

Our model can be regarded as an interacting multi-species particle system on the ring. We interpret the local state $k_i = \alpha$ as representing the site i occupied by a particle of the αth kind. The transition (2.1) is viewed as a local hopping process of particles. We identify the first kind of particles with vacancies. Note that a particle α has been assumed to overtake any $\beta (< \alpha)$ with the same rate p as the vacancy $\beta = 1$.

We call this model the multi-species ASEP or more specifically the $(N-1)$-species ASEP. The usual ASEP corresponds to $N = 2$. We will be formally concerned with the zero-species ASEP $(N = 1)$ as well. The case $p = q$ will be called the multi-species symmetric simple exclusion process (SSEP).

Let $|1\rangle, \dots, |N\rangle$ be the basis of the single-site space $\mathbb{C}^N$ and represent a particle configuration $(k_1, \dots, k_L)$ as the ket vector $|k_1, \dots, k_L\rangle = |k_1\rangle \otimes \cdots \otimes |k_L\rangle \in (\mathbb{C}^N)^{\otimes L}$. In terms of the probability vector

$$|P(t)\rangle = \sum_{1 \leqslant k_i \leqslant N} P(k_1, \dots, k_L; t) |k_1, \dots, k_L\rangle, \tag{2.4}$$

the master equation (2.2) is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = H|P(t)\rangle, \tag{2.5}$$

where the linear operator H has the form

$$H = \sum_{i\in\mathbb{Z}_L} h_{i,i+1}, \tag{2.6}$$

$$h = \sum_{1\leqslant\alpha<\beta\leqslant N} (-pE_{\beta\beta}\otimes E_{\alpha\alpha} - qE_{\alpha\alpha}\otimes E_{\beta\beta} + pE_{\alpha\beta}\otimes E_{\beta\alpha} + qE_{\beta\alpha}\otimes E_{\alpha\beta}). \tag{2.7}$$

Here $h_{i,i+1}$ acts on the ith and the $(i+1)$th components of the tensor product as h and as the identity elsewhere. $E_{\alpha\beta}$ denotes the $N \times N$ matrix unit sending $|\gamma\rangle$ to $\delta_{\gamma\beta}|\alpha\rangle$.

Equation (2.5) has the form of the Schrödinger equation with imaginary time and thus provides our multi-species ASEP with a quantum Hamiltonian formalism. Of course in the present case, $P(k_1,\ldots,k_L;t)$ itself gives the probability distribution unlike the squared wavefunctions in the case of quantum mechanics. Nevertheless we call the matrix H the *Hamiltonian*[4] in this paper by the abuse of language.

2.2. Basic properties of Hamiltonian

Our Hamiltonian H is an $N^L \times N^L$ matrix whose off-diagonal elements are p, q or 0, and diagonal elements belong to $p\mathbb{Z}_{\leqslant 0} + q\mathbb{Z}_{\leqslant 0}$. Each column of H sums up to 0 assuring the conservation of the total probability $\sum_{1\leqslant k_i\leqslant N} P(k_1,\ldots,k_L;t)$. It enjoys the symmetries

$$[H, C] = 0, \qquad RHR^{-1} = QHQ^{-1} = H|_{p\leftrightarrow q}, \tag{2.8}$$

where C, R and Q are linear operators defined by

$$C|k_1,\ldots,k_L\rangle = |k_L,k_1,\ldots,k_{L-1}\rangle \quad \text{(cyclic shift)}, \tag{2.9}$$

$$R|k_1,\ldots,k_L\rangle = |k_L,k_{L-1},\ldots,k_1\rangle \quad \text{(reflection)}, \tag{2.10}$$

$$Q|k_1,\ldots,k_L\rangle = |N+1-k_1,\ldots,N+1-k_L\rangle \quad \text{(`charge conjugation')} \tag{2.11}$$

satisfying $C^L = R^2 = Q^2 = 1$. H is Hermitian only at $p = q$, where it becomes the Hamiltonian of the $sl(N)$-invariant Heisenberg spin chain $H = p\sum_{i\in\mathbb{Z}_L}(\sigma_{i,i+1} - 1)$ with $\sigma_{i,i+1}$ being the transposition of the local states $k_i \leftrightarrow k_{i+1}$. In general, imaginary eigenvalues of H form complex conjugate pairs. In section 5, Bethe ansatz integrability of H for general p, q will be demonstrated. Although H is not normal in general, we expect that it is diagonalizable based on conjecture 5.1 and the remark following it. This fact will be used only in corollary 4.8.

In view of the transition rule (2.1), our Hamiltonian H obviously preserves the number of particles of each kind. It follows that the space of states $(\mathbb{C}^N)^{\otimes L}$ splits into a direct sum of sectors and H has the block diagonal structure:

$$(\mathbb{C}^N)^{\otimes L} = \bigoplus_m V(m), \qquad V(m) = \bigoplus_{\{k_i\}\,\text{in sector}\,m} \mathbb{C}|k_1,\ldots,k_L\rangle, \tag{2.12}$$

$$H = \bigoplus_m H(m), \qquad H(m) \in \mathrm{End}\,V(m). \tag{2.13}$$

[4] A more proper terminology is the Markov matrix.

Here the direct sums $\bigoplus_m$ extend over $m = (m_1, \ldots, m_N) \in \mathbb{Z}_{\geqslant 0}^N$ such that $m_1 + \cdots + m_N = L$. The array m labels a sector $V(m)$, which is a subspace of $(\mathbb{C}^N)^{\otimes L}$ specified by the multiplicities of particles of each kind. Namely, the latter sum in (2.12) is taken over all $(k_1, \ldots, k_L) \in \{1, \ldots, N\}^L$ such that

$$\mathrm{Sort}(k_1, \ldots, k_L) = \overbrace{1 \ldots 1}^{m_1}\overbrace{2 \ldots 2}^{m_2} \cdots \overbrace{N \ldots N}^{m_N}, \tag{2.14}$$

where Sort stands for the ordering non-decreasing to the right. In other words, the $\{k_i\}$ in (2.12) runs over all the permutations of the right-hand side of (2.14). The array m itself will also be called a sector. A sector, $m = (2, 0, 4, 1, 0, 3)$ for instance, will also be referred in terms of the Sort sequence as $1^2 3^4 4 6^3 = 1133334666$. By the definition $\dim V(m) = L\big/\prod_{i=1}^N m_i!$.

By now we have separated the master equation (2.5) into those in each sector $\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = H(m)|P(t)\rangle$ with $|P(t)\rangle \in V(m)$. Since the transition rule (2.1) only refers to the alternatives $\alpha > \beta$ or $\alpha < \beta$, the master equation in the sector $1^2 3^4 4 6^3$ for example is equivalent to that in $a^2 b^4 c d^3$ for any $1 \leqslant a < b < c < d \leqslant N$.

Henceforth without loss of generality we take $N = L$ and restrict our consideration to the *basic sectors* that have the form $m = (m_1, \ldots, m_n, 0, \ldots, 0)$ for some n with $m_1, \ldots, m_n$ being all positive. The basic sectors are labeled with the elements of the set

$$\mathcal{M} = \left\{(m_1, \ldots, m_n) \in \mathbb{Z}_{\geqslant 1}^n \,\middle|\, 1 \leqslant n \leqslant L, m_1 + \cdots + m_n = L\right\}. \tag{2.15}$$

In this convention, which will be employed in the rest of the paper except section 5, n plays the role of N in the sense that $H(m)$ for $m = (m_1, \ldots, m_n) \in \mathcal{M}$ is equivalent to the Hamiltonian of the $(n-1)$-species ASEP.

In section 3 we study specific eigenvalues of H that are relevant to the leading behavior of the relaxation. In section 4 we elucidate a spectral duality, a new global aspect of the spectrum of H, which has escaped a notice in earlier works mostly devoted to the studies of the thermodynamic limit $L \to \infty$ under a fixed N.

3. Relaxation to the stationary state

3.1. General remarks

The initial value problem of the master equation $\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = H(m)|P(t)\rangle$ with $|P(t)\rangle \in V(m)$ in the sector $m = (m_1, \ldots, m_n) \in \mathcal{M}$ (2.15) is formally solved as

$$|P(t)\rangle = \mathrm{e}^{tH(m)}|P(0)\rangle. \tag{3.1}$$

There is a unique stationary state corresponding to the zero eigenvalue of $H(m)$. The stationary state has a non-uniform probability distribution [PEM] except for the zero-species, the one-species and the SSEP cases, where $P(k_1, \ldots, k_L) = 1/\dim V(m)$. All the other eigenvalues of $H(m)$ have strictly negative real parts, which are responsible for various relaxation modes to the stationary state. (The associated eigenvectors themselves are not physical probability vectors having non-negative components.) Let us denote by $|P(\infty)\rangle$ the stationary state. In general, the system exhibits the long-time behavior

$$|P(t)\rangle - |P(\infty)\rangle \sim \mathrm{e}^{-t/\tau} \quad (t \to \infty), \tag{3.2}$$

where τ is the relaxation time. An important characteristic of the non-equilibrium dynamics is the scaling property of the relaxation time with respect to the system size:

$$\tau \sim L^z \quad (L \to \infty), \tag{3.3}$$

where z is the dynamical exponent. The thermodynamic limit $L \to \infty$ is to be taken under the fixed densities $\rho_j = m_j/L$ for $j = 1, \ldots, n$. (Recall $L = m_1 + \cdots + m_n$ in (2.15), and therefore $\rho_1 + \cdots + \rho_n = 1$.)

The spectrum in a sector $m \in \mathcal{M}$ is denoted by

$$\mathrm{Spec}(m) = \text{multiset of eigenvalues of } H(m), \tag{3.4}$$

where the multiplicities count the degrees of degeneracy. Spec(m) is invariant under complex conjugation. The charge conjugation property (2.11) implies the symmetry

$$\mathrm{Spec}(m_1, \ldots, m_n) = \mathrm{Spec}(m_n, \ldots, m_1). \tag{3.5}$$

We say that a complex eigenvalue x of $H(m)$ is larger (smaller) than y if $\mathrm{Re}(x) > \mathrm{Re}(y)$ ($\mathrm{Re}(x) < \mathrm{Re}(y)$). Spec$(m)$ contains 0 as the unique largest eigenvalue. For a finite L, we say an eigenvalue E of $H(m)$ is *second largest* if

$$\mathrm{Re}\, E = \max \mathrm{Re}(\mathrm{Spec}(m)\backslash\{0\}). \tag{3.6}$$

From (3.1) and (3.2), the scaling property (3.3) is equivalent to the following behavior of the second largest eigenvalues

$$\max \mathrm{Re}(\mathrm{Spec}(m)\backslash\{0\}) = -cL^{-z} + o(L^{-z}) \quad (L \to \infty) \tag{3.7}$$

if the initial condition $|P(0)\rangle$ is generic. Here $c > 0$ is an 'amplitude' which can depend on $\rho_1, \ldots, \rho_n$ in general, but not on L.

In the remainder of this section, we derive the exponent z of the multi-species ASEP based on (3.7). Our argument reduces the problem essentially to the one-species case and is partly based on a conjecture supported by numerical analyses.

3.2. Known results on the one-species ASEP

In this subsection we review the known results on the one-species ASEP. Thus we shall exclusively consider the sector of the form $m = (m_1, m_2) \in \mathcal{M}$, and regard the local states 1 and 2 as vacancies and the particles of one kind, respectively. Recall also that $L = m_1 + m_2$.

The second largest eigenvalues are known to form a complex-conjugate pair, which will be denoted by $E^{\pm}(m)$. See figure 1. When $m_1 = m_2$ or $p = q$, the degeneracy $E^{+}(m) = E^{-}(m) \in \mathbb{R}$ occurs.

In [GS, K, GM], the large L asymptotic form

$$E^{\pm}((1-\rho)L, \rho L) = \pm 2\pi \mathrm{i}|(p-q)(1-2\rho)|L^{-1} - 2C|p-q|\sqrt{\rho(1-\rho)}L^{-\frac{3}{2}} + O(L^{-2}) \tag{3.8}$$

with a fixed particle density $\rho = m_2/L$ was derived for $p \neq q$ by an analysis of the Bethe equation. The two terms are both invariant under $\rho \leftrightarrow 1-\rho$. The constant C has been numerically evaluated as $C = 6.509\,189\,337\,94\ldots$. Thus from (3.7) the one-species ASEP for $p \neq q$ has the dynamical exponent $z = \frac{3}{2}$, which is a characteristic value for the Kardar–Parisi–Zhang universality class [KPZ].

In the SSEP case $p = q$, the Hamiltonian $H(m_1, m_2)$ is Hermitian, and hence all the eigenvalues are real. The system relaxes to the equilibrium stationary state. For a finite L, the second largest eigenvalues take the simple form

$$E^{+}(m_1, m_2) = E^{-}(m_1, m_2) = -4p\sin^2\left(\frac{\pi}{L}\right) \quad (0 < m_2 < L), \tag{3.9}$$

which is independent of the density $\rho = m_2/L$ as long as $0 < \rho < 1$. The asymptotic behavior in $L \to \infty$ is easily determined as

$$E^{\pm}((1-\rho)L, \rho L) = -4\pi^2 pL^{-2} + O(L^{-4}), \tag{3.10}$$

which is free from a contribution of order $L^{-\frac{3}{2}}$. From (3.10), we find the dynamical exponent $z = 2$, which is the characteristic value for the Edwards–Wilkinson universality class [EW].

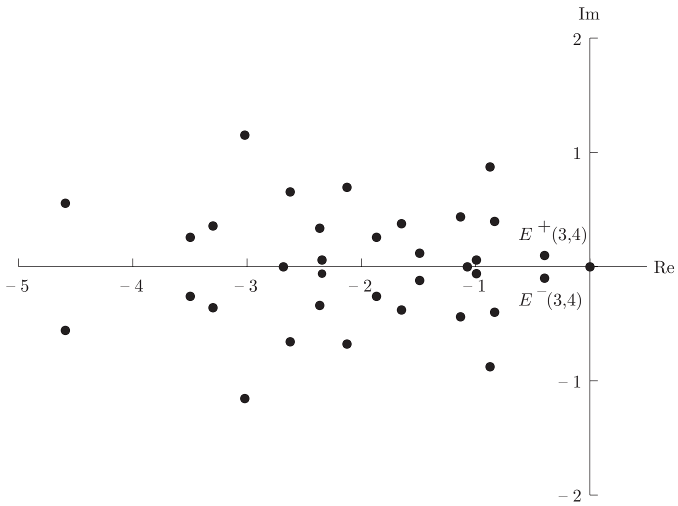

Figure 1. Spec(3, 4) for $(p, q) = (0.8, 0.2)$.

3.3. *Eigenvalues of* $H(2, 1, 3, 1)$*: an example*

Let us proceed to the multi-species ASEP. Before considering a general sector in the next subsection, we illustrate characteristic features of the spectrum along an example. Figure 2(a) is a plot (•) of the spectrum Spec(2, 1, 3, 1) on the complex plane. We recall that the sector (2, 1, 3, 1) means the ring of length 7 populated with four kinds of particles with multiplicities 2, 1, 3 and 1, among which the first kind ones are regarded as vacancies.

For comparison, we have also included the plot of the spectra in the one-species sectors (2, 5), (3, 4) and (6, 1) in different shapes. These one-species sectors are related to the multi-species sector (2, 1, 3, 1) as follows. The sector $(2, 5) = (2, 1 + 3 + 1)$ is obtained by the identification of all kinds of particles (except for vacancies) as one kind of particles. The sector $(3, 4) = (2 + 1, 3 + 1)$ is obtained by the identification of the second kind particles as vacancies and the rest of the particles as one kind of particles. The sector $(6, 1) = (2+1+3, 1)$ is obtained by identification of the second and the third kinds of particles as vacancies. In figure 2(*a*), we observe that all ×, □ and Δ overlap •. Namely, Spec(2, 5), Spec(3, 4), Spec(6, 1) are totally embedded into Spec(2, 1, 3, 1).

Figure 2(*b*) shows that the second largest eigenvalues (denoted by $E_j^{\pm}(2, 1, 3, 1)$) in those one-species sectors form a string within Spec(2, 1, 3, 1) near the origin. Although their real parts are not strictly the same, there is no • between the string and the origin. More precisely, there is no eigenvalue in the sector (2, 1, 3, 1) which is nonzero and larger than any second largest eigenvalues in the one-species sectors (2, 5), (3, 4) and (6, 1). This property is a key to our argument in the following.

3.4. *Eigenvalues of multi-species ASEP*

Let us systematize the observations made in the previous subsection. First we claim that the following inclusion relation holds generally:

$$\mathrm{Spec}(m_1, \ldots, m_n) \supset \mathrm{Spec}(m_1 + \cdots + m_j, m_{j+1} + \cdots + m_n) \quad (1 \leqslant j \leqslant n - 1). \tag{3.11}$$

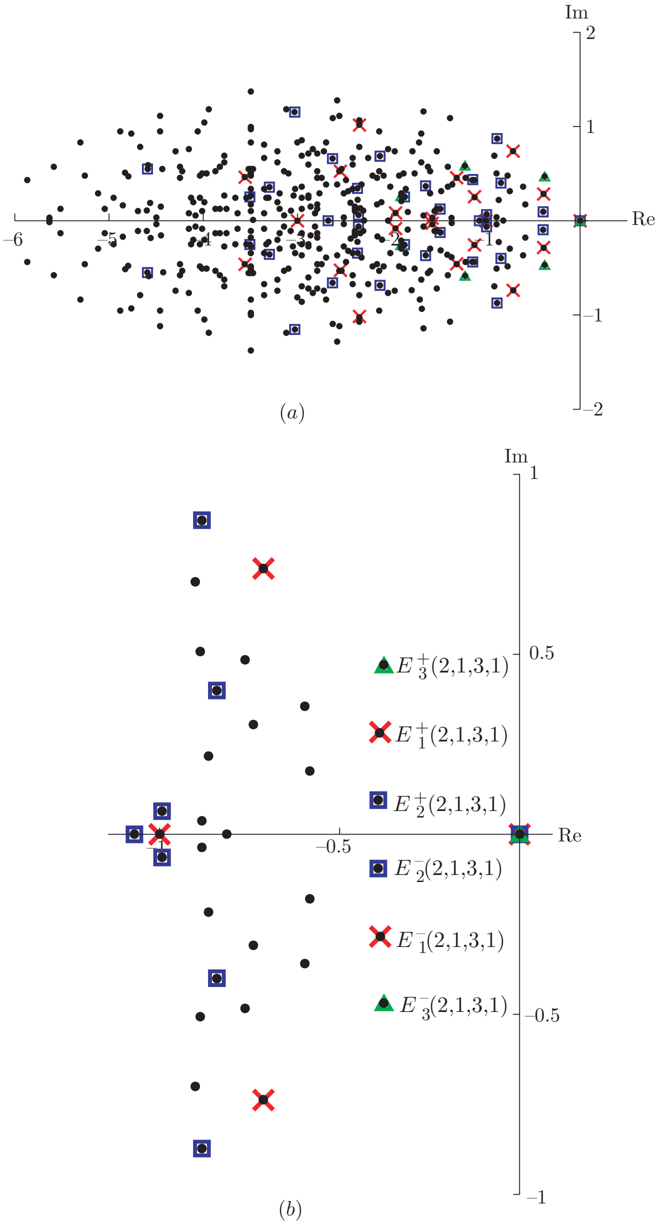

Figure 2. (*a*) Spec(2, 1, 3, 1) (•), Spec(2, 5) (×), Spec(3, 4) (□) and Spec(6, 1) (Δ) with $(p, q) = (0.8, 0.2)$. (*b*) An enlarged view near the origin.

Each one-species sector appearing in the right-hand side of (3.11) is obtained by the identification similar to the previous subsection:

$$\underbrace{\overbrace{\circ\cdots\circ}^{m_1}|\cdots|\overbrace{\circ\cdots\circ}^{m_j}}_{m_1+\cdots+m_j}|\underbrace{\overbrace{\circ\cdots\circ}^{m_{j+1}}|\cdots|\overbrace{\circ\cdots\circ}^{m_n}}_{m_{j+1}+\cdots+m_n}.$$

The relation (3.11) is a special case of the more general statement in theorem 4.5. See also section 5.2.1 for an account from the nested Bethe ansatz.

Next we introduce a class of eigenvalues of $H(m)$ for a multi-species sector $m = (m_1, \dots, m_n) \in \mathcal{M}$ by

$$E_j^{\pm}(m) = E^{\pm}(m_1 + \cdots + m_j, m_{j+1} + \cdots + m_n) \quad (1 \leqslant j \leqslant n-1), \tag{3.12}$$

where $E^{\pm}$ in the right-hand side are the second largest eigenvalues in the one-species sector introduced in section 3.2. In view of (3.11), we know $E_j^{\pm}(m) \in \mathrm{Spec}(m)$. Note that $\mathrm{Re}\big(E_j^{+}(m)\big) = \mathrm{Re}\big(E_j^{-}(m)\big)$, but the subscript j does not necessarily reflect the ordering of the eigenvalues with respect to their real parts. Generalizing the previous observation in figure 2, we make the following conjecture.

Conjecture 3.1. *In any sector* $m = (m_1, \dots, m_n) \in \mathcal{M}$*, there is no eigenvalue* $E \in \mathrm{Spec}(m)$ *such that*

$$\max\left\{\mathrm{Re}\, E_1^{\pm}(m), \dots, \mathrm{Re}\, E_{n-1}^{\pm}(m)\right\} < \mathrm{Re}\, E < 0. \tag{3.13}$$

The one-species case $n = 2$ is trivially true by the definition. So far, for $n \geqslant 3$, the conjecture has been checked in all the basic sectors m satisfying $\dim V(m) < 8000$ at $(p, q) = (1, 0), (0.9, 0.1), (0.8, 0.2), (0.7, 0.3)$ and $(0.6, 0.4)$. The number of such sectors is 891 and the largest L with L different species of particles is 7. ($\dim V(1, 1, 1, 1, 1, 1, 1) = 5040$.)

Admitting the conjecture, we are able to claim that the second largest eigenvalues in $\mathrm{Spec}(m)$ are equal to $E_j^{\pm}(m)$ for some $1 \leqslant j \leqslant n-1$. (Such j may not be unique.) The asymptotic behavior of $E_j^{\pm}(m)$ is derived from (3.8) and (3.12) as

$$E_j^{\pm}(\rho_1 L, \dots, \rho_n L) = \pm 2\pi \mathrm{i} |(p-q)(1-2r_j)| L^{-1} - 2C|p-q|\sqrt{r_j(1-r_j)}\, L^{-\frac{3}{2}} + O(L^{-2}) \tag{3.14}$$

for $p \neq q$, where $r_j = \rho_1 + \cdots + \rho_j$ is fixed. We remark that the leading terms in (3.14) depend on j only through r_j in the amplitudes. We call the eigenvalues $E_1^{\pm}(m), \dots, E_{n-1}^{\pm}(m)$ *next leading*. Thus the second largest eigenvalues are next leading. All the next leading eigenvalues possess the same asymptotic behavior as the second largest ones up to the amplitudes as far as the first two leading terms in (3.14) are concerned.

With regard to the SSEP case $p = q$, the stationary state is an equilibrium state. $H(m)$ is Hermitian and $E_j^{\pm}(m)$ is real. We have the following explicit form as in the one-species case:

$$E_1^{\pm}(m) = \cdots = E_{n-1}^{\pm}(m) = -4p \sin^2\left(\frac{\pi}{L}\right). \tag{3.15}$$

In other words, the next leading eigenvalues $E_j^{\pm}(m)$ are degenerated in the SSEP limit $p - q \to 0$. See figure 3, where the string of the next leading eigenvalues shrinks to a point on the real axis as $p - q$ approaches 0.

As the one-species case (3.10), we find

$$E_j^{\pm}(\rho_1 L, \dots, \rho_n L) = -4\pi^2 p L^{-2} + O(L^{-4}). \tag{3.16}$$

To summarize, the results (3.14) and (3.16) lead to the following behavior of the relaxation time τ:

$$\tau \sim \begin{cases} L^{\frac{3}{2}} & \text{for} \quad p \neq q, \\ L^2 & \text{for} \quad p = q, \end{cases} \qquad (L \to \infty). \tag{3.17}$$

Therefore we conclude that the dynamical exponent of the multi-species ASEP is independent of the number of species. It belongs to the KPZ universality class $\left(z = \frac{3}{2}\right)$ for $p \neq q$, and to the EW universality class $(z = 2)$ for $p = q$.

We leave it for a future study to investigate the gap between the next leading eigenvalues and further smaller eigenvalues, which governs the pre-asymptotic behavior of the multi-species ASEP.

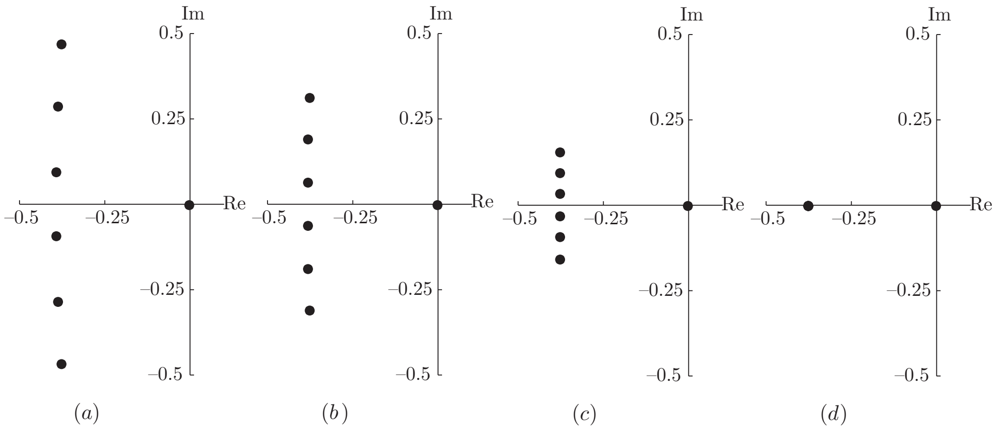

Figure 3. Degeneracy in the SSEP limit $p - q \to 0$ in the sector $(2, 1, 3, 1)$. (p, q) is taken as (*a*) (0.8, 0.2), (*b*) (0.7, 0.3), (*c*) (0.6, 0.4) and (*d*) (0.5, 0.5).

4. Duality in the spectrum

Throughout this section, a sector means a basic sector as explained in section 2.2. We fix the number of sites in the ring $L \in \mathbb{Z}_{\geqslant 2}$. Our goal is to prove theorem 4.12, which exhibits a duality in the spectrum of the Hamiltonian.

4.1. Another label of sectors

Set

$$\Omega = \{1, 2, \ldots, L-1\}, \tag{4.1}$$

$$\mathcal{S} = \text{the power set (i.e. the set of all subsets of) } \Omega. \tag{4.2}$$

Recall that the sectors in the length L chain are labeled with the set $\mathcal{M}$ (2.15). We identify $\mathcal{M}$ with $\mathcal{S}$ by the one to one correspondence:

$$\mathcal{M} \ni m = (m_1, \ldots, m_n) \longleftrightarrow \{s_1 < \cdots < s_{n-1}\} = \mathfrak{s} \in \mathcal{S} \tag{4.3}$$

specified via $s_j = m_1 + m_2 + \cdots + m_j$, namely,

$$\underbrace{\underbrace{\overbrace{\underbrace{\circ \cdots \circ}_{s_1}}^{m_1} | \overbrace{\circ \cdots \circ}^{m_2}}_{s_2} | \cdots |}_{s_{n-1}} \overbrace{\circ \cdots \circ}^{m_n},$$

where the numbers of the symbols $\circ$ and $|$ are L and $n-1$, respectively. For example the identification $\mathcal{M} \leftrightarrow \mathcal{S}$ for $L = 4$ is given as follows:

$$(1, 1, 1, 1) \leftrightarrow \{1, 2, 3\}$$

$$(1, 1, 2) \leftrightarrow \{1, 2\} \quad (1, 2, 1) \leftrightarrow \{1, 3\} \quad (2, 1, 1) \leftrightarrow \{2, 3\}$$

$$(1, 3) \leftrightarrow \{1\} \quad (2, 2) \leftrightarrow \{2\} \quad (3, 1) \leftrightarrow \{3\}$$

$$(4) \leftrightarrow \emptyset.$$

An element of $\mathcal{S}$ will also be called a sector. In the remainder of this section we will mostly work with the label $\mathcal{S}$ instead of $\mathcal{M}$. There are $\sharp\mathcal{S} = 2^{L-1}$ distinct sectors. We employ the notation:

$$\bar{\mathfrak{s}} = \Omega\backslash\mathfrak{s} = \text{complement sector of } \mathfrak{s}. \tag{4.4}$$

For a sector $\mathfrak{s} = \{s_1 < \cdots < s_{n-1}\} \in \mathcal{S}$, we introduce the set $\mathcal{P}(\mathfrak{s})$ by (see (2.14)

$$\mathcal{P}(\mathfrak{s}) = \{k = (k_1, \ldots, k_L) \mid \text{Sort}(k) = \overbrace{1\ldots1}^{s_1}\overbrace{2\ldots2}^{s_2-s_1}\cdots\overbrace{n-1\ldots n-1}^{s_{n-1}-s_{n-2}}\overbrace{n\ldots n}^{L-s_{n-1}}\}, \tag{4.5}$$

where Sort stands for the ordering non-decreasing to the right as in (2.14). For $\mathfrak{s} = \emptyset \in \mathcal{S}$, this definition should be understood as $\mathcal{P}(\emptyset) = \{(1, \ldots, 1)\}$.

To each sector we associate the bra and ket vector spaces

$$V^*_{\mathfrak{s}} = \bigoplus_{k\in\mathcal{P}(\mathfrak{s})} \mathbb{C}\langle k_1, \ldots, k_L|, \qquad V_{\mathfrak{s}} = \bigoplus_{k\in\mathcal{P}(\mathfrak{s})} \mathbb{C}|k_1, \ldots, k_L\rangle. \tag{4.6}$$

Here $k_1, \ldots, k_L \in \{1, \ldots, L\}$ stand for local states. For example if $L = 3$, one has

$$\begin{aligned}
&V_\emptyset = \mathbb{C}|111\rangle,\\
&V_{\{1\}} = \mathbb{C}|122\rangle \oplus \mathbb{C}|212\rangle \oplus \mathbb{C}|221\rangle,\\
&V_{\{2\}} = \mathbb{C}|112\rangle \oplus \mathbb{C}|121\rangle \oplus \mathbb{C}|211\rangle,\\
&V_\Omega = V_{\{1,2\}} = \mathbb{C}|123\rangle \oplus \mathbb{C}|132\rangle \oplus \mathbb{C}|213\rangle \oplus \mathbb{C}|231\rangle \oplus \mathbb{C}|312\rangle \oplus \mathbb{C}|321\rangle.
\end{aligned}$$

Note that the vectors such as $|222\rangle$ and $|113\rangle$ are *not* included in any $V_{\mathfrak{s}}$ because we are concerned with basic sectors only (see (4.5)). In general, one has

$$\dim V_{\mathfrak{s}} = \dim V^*_{\mathfrak{s}} = \frac{L!}{s_1!(s_2 - s_1)! \cdots (s_{n-1} - s_{n-2})!(L - s_{n-1})!} \tag{4.7}$$

for $\mathfrak{s} = \{s_1 < \cdots < s_{n-1}\} \in \mathcal{S}$.

Suppose $\mathcal{M} \ni m \leftrightarrow \mathfrak{s} \in \mathcal{S}$ under the correspondence (4.3). We renew the symbols $H(m)$ and $V(m)$ in (2.12) and (2.13) as

$$V_{\mathfrak{s}} = V(m), \qquad H_{\mathfrak{s}} = H(m). \tag{4.8}$$

The set $\mathcal{S}$ is equipped with the natural poset (partially ordered set) structure with respect to $\subseteq$. The poset structure is encoded in the Hasse diagram [St], which is useful in our working below. In the present case, it is just the $(L - 1)$-dimensional hypercube, where each vertex corresponds to a sector. Sectors are so arranged that every edge of the hypercube becomes an arrow $\mathfrak{s} \to \mathfrak{t}$ meaning that $\mathfrak{s} \subset \mathfrak{t}$ and $\sharp\mathfrak{t} = \sharp\mathfrak{s} + 1$. There is the unique sink corresponding to the maximal sector $\Omega \in \mathcal{S}$ and the unique source corresponding to the minimal sector $\emptyset \in \mathcal{S}$ (see figure 4).

We introduce the natural bilinear pairing between the bra and ket vectors by

$$\langle k_1, \ldots, k_L | j_1, \ldots, j_L\rangle = \delta_{k_1,j_1} \cdots \delta_{k_L,j_L}. \tag{4.9}$$

With respect to the pairing, $V^*_{\mathfrak{s}}$ and $V_{\mathfrak{t}}$ are dual if $\mathfrak{s} = \mathfrak{t}$ and orthogonal if $\mathfrak{s} \neq \mathfrak{t}$.

Any linear operator $\overrightarrow{G}$ acting on ket vectors gives rise to the unique linear operator $\overleftarrow{G}$ acting on bra vectors via $(\langle k_1, \ldots, k_L|\overleftarrow{G})|j_1, \ldots, j_L\rangle = \langle k_1, \ldots, k_L|(\overrightarrow{G}|j_1, \ldots, j_L\rangle)$ and vice versa. We write this quantity simply as $\langle k_1, \ldots, k_L|G|j_1, \ldots, j_L\rangle$ as usual, and omit $\leftarrow$ and $\rightarrow$ unless an emphasis is preferable. The transpose G^{T} of G is defined by $\langle k|G^{\mathrm{T}}|j\rangle = \langle j|G|k\rangle (=: G_{jk})$ for any $k = (k_1, \ldots, k_L)$ and $j = (j_1, \ldots, j_L)$. Of course $\overrightarrow{G}^{\mathrm{T}}$ is equivalent to $\overleftarrow{G}$ in the sense that

$$G^{\mathrm{T}}|k\rangle = \sum_j G_{kj}|j\rangle, \qquad \langle k|G = \sum_j G_{kj}\langle j|. \tag{4.10}$$

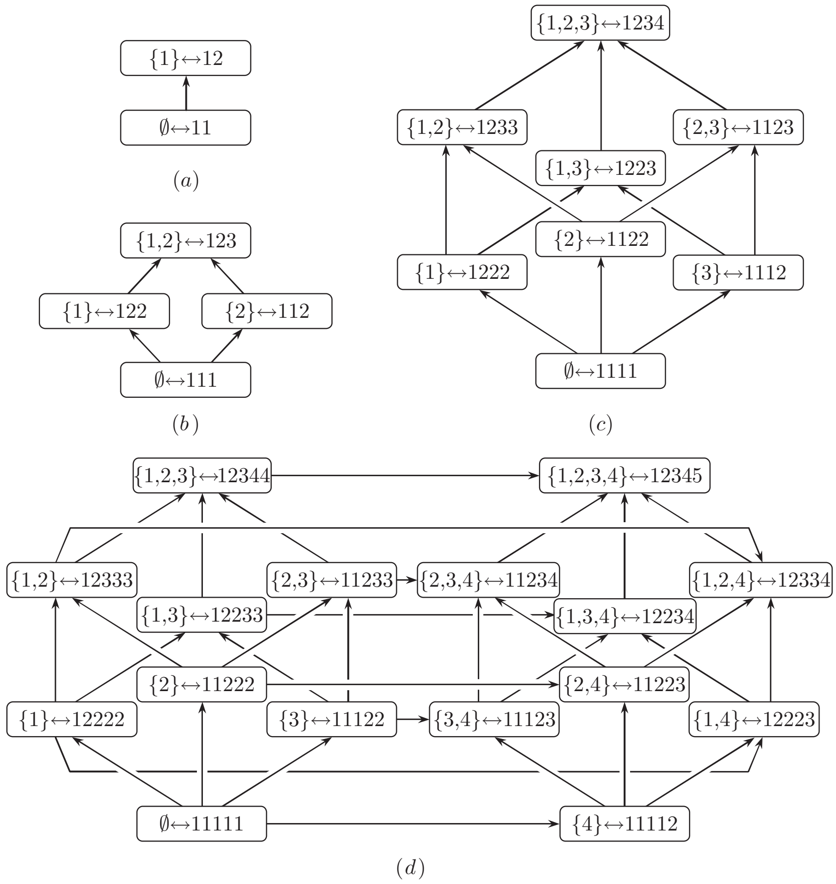

Figure 4. Hasse diagrams (*a*), (*b*), (*c*) and (*d*) for $L = 2, 3, 4$ and 5, respectively. Sectors are labeled by $\mathcal{S}$ (4.2) as well as the sequence Sort(k) of local states as in (4.5).

4.2. *Operator* $\varphi_{\mathfrak{s},\mathfrak{t}}$

Let $\mathfrak{s}, \mathfrak{t} \in \mathcal{S}$ be sectors such that $\mathfrak{s} \subseteq \mathfrak{t}$. We introduce a $\mathbb{C}$-linear operator $\varphi_{\mathfrak{s},\mathfrak{t}}$ in terms of its action on ket vectors $\overrightarrow{\varphi}_{\mathfrak{s},\mathfrak{t}} : V_{\mathfrak{t}} \to V_{\mathfrak{s}}$. We define $\varphi_{\mathfrak{s},\mathfrak{s}}$ to be the identity operator for any sector $\mathfrak{s}$. Before giving the general definition of the case $\mathfrak{s} \subset \mathfrak{t}$, we illustrate it with the example $\mathfrak{s} = \{2, 5\} \subset \mathfrak{t} = \{2, 3, 5, 8\}$ with $L = 9$. The Sort sequence of the local states in the sense of (4.5) for $\mathcal{P}(\mathfrak{t})$ and $\mathcal{P}(\mathfrak{s})$ read as follows:

$$\begin{aligned} &\mathcal{P}(\mathfrak{t} = \{2, 3, 5, 8\}) : \quad 11\overset{2}{|}2\overset{3}{|}33\overset{5}{|}444\overset{8}{|}5, \\ &\mathcal{P}(\mathfrak{s} = \{2, 5\}) : \quad 11\overset{2}{|}222\overset{5}{|}3333. \end{aligned} \tag{4.11}$$

According to these lists, we define $\varphi_{\mathfrak{s},\mathfrak{t}}$ to be the operator replacing the local states as $3 \to 2, 4 \to 3, 5 \to 3$ (keeping 1 and 2 unchanged) within all the ket vectors $|k_1, \ldots, k_L\rangle$ in $V_{\mathfrak{t}}$.

The general definition of $\varphi_{\mathfrak{s},\mathfrak{t}}$ is similar and goes as follows. Suppose $\mathfrak{t} = \{t_1 < \cdots < t_n\}$ and $\mathfrak{s} = \mathfrak{t}\backslash\{t_{i_1}, \ldots, t_{i_l}\}$. Then $\varphi_{\mathfrak{s},\mathfrak{t}}$ is a $\mathbb{C}$-linear operator determined by its action on base vectors as follows:

$$\begin{array}{rccc} \vec{\varphi}_{\mathfrak{s},\mathfrak{t}}: & V_{\mathfrak{t}} & \longrightarrow & V_{\mathfrak{s}} \\ & |k_1, \ldots, k_L\rangle & \mapsto & |k_1', \ldots, k_L'\rangle, \end{array} \tag{4.12}$$

where $x' = x - \sharp\{i_j \mid i_j < x\}$.

Example 4.1.

$|\phi\rangle := |21433\rangle - |12343\rangle \in V_{\mathfrak{t}}\ (\mathfrak{t} = \{1, 2, 4\}, L = 5)$,

$$\varphi_{12,124}|\phi\rangle = |21333\rangle - |12333\rangle, \qquad \begin{cases} \varphi_{1,12}\varphi_{12,124}|\phi\rangle = |21222\rangle - |12222\rangle, \\ \varphi_{2,12}\varphi_{12,124}|\phi\rangle = 0, \end{cases}$$

$$\varphi_{14,124}|\phi\rangle = |21322\rangle - |12232\rangle, \qquad \begin{cases} \varphi_{1,14}\varphi_{14,124}|\phi\rangle = |21222\rangle - |12222\rangle, \\ \varphi_{4,14}\varphi_{14,124}|\phi\rangle = |11211\rangle - |11121\rangle, \end{cases}$$

$$\varphi_{24,124}|\phi\rangle = |11322\rangle - |11232\rangle, \qquad \begin{cases} \varphi_{2,24}\varphi_{24,124}|\phi\rangle = 0, \\ \varphi_{4,24}\varphi_{24,124}|\phi\rangle = |11211\rangle - |11121\rangle, \end{cases}$$

$\varphi_{1,124}|\phi\rangle = |21222\rangle - |12222\rangle$,
$\varphi_{4,124}|\phi\rangle = |11211\rangle - |11121\rangle$,
$\varphi_{2,124}|\phi\rangle = 0, \quad \varphi_{\emptyset,124}|\phi\rangle = 0$,

where $\varphi_{12,124}$ is an abbreviation of $\varphi_{\{1,2\},\{1,2,4\}}$, etc.

The following property of $\varphi_{\mathfrak{s},\mathfrak{t}}$ is a direct consequence of the definition.

Lemma 4.2. *For a pair of sectors $\mathfrak{s} \subset \mathfrak{t}$, let $\mathfrak{s}_0 \subset \mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_l$ be any sectors such that $\mathfrak{s}_0 = \mathfrak{s}$, $\mathfrak{s}_l = \mathfrak{t}$ and $\sharp\mathfrak{s}_{j+1} = \sharp\mathfrak{s}_j + 1$ for all $0 \leqslant j < l$. Then,*

$$\varphi_{\mathfrak{s},\mathfrak{t}} = \varphi_{\mathfrak{s},\mathfrak{s}_1}\varphi_{\mathfrak{s}_1,\mathfrak{s}_2}\cdots\varphi_{\mathfrak{s}_{l-1},\mathfrak{t}}.$$

In particular, the composition in the right-hand side is independent of the choice of the intermediate sectors $\mathfrak{s}_1, \ldots, \mathfrak{s}_{l-1}$.

In example 4.1, one can observe, for instance, $\varphi_{4,124}|\phi\rangle = \varphi_{4,14}\varphi_{14,124}|\phi\rangle = \varphi_{4,24}\varphi_{24,124}|\phi\rangle$.

Let us turn to the transpose $\varphi_{\mathfrak{s},\mathfrak{t}}^{\mathrm{T}}$. By the definition (see (4.10)), we have

$$\begin{array}{rccc} \vec{\varphi}_{\mathfrak{s},\mathfrak{t}}^{\mathrm{T}}: & V_{\mathfrak{s}} & \longrightarrow & V_{\mathfrak{t}} \\ & |k_1, \ldots, k_L\rangle & \mapsto & \displaystyle\sum_{j\in\mathcal{P}(\mathfrak{t})}{}' |j_1, \ldots, j_L\rangle, \end{array} \tag{4.13}$$

where Σ' extends over those $j = (j_1, \ldots, j_L) \in \mathcal{P}(\mathfrak{t})$ such that $\varphi_{\mathfrak{s},\mathfrak{t}}|j_1, \ldots, j_L\rangle = |k_1, \ldots, k_L\rangle$. For example in example 4.1, one has

$$\varphi_{14,124}^{\mathrm{T}}|21322\rangle = |21433\rangle + |31423\rangle + |31432\rangle.$$

From (4.13) and (4.12) it follows that $\varphi_{\mathfrak{s},\mathfrak{t}}\varphi_{\mathfrak{s},\mathfrak{t}}^{\mathrm{T}} = \frac{\dim V_{\mathfrak{t}}}{\dim V_{\mathfrak{s}}}\,\mathrm{Id}$, which actually means

$$\vec{\varphi}_{\mathfrak{s},\mathfrak{t}}\vec{\varphi}_{\mathfrak{s},\mathfrak{t}}^{\mathrm{T}} = \frac{\dim V_{\mathfrak{t}}}{\dim V_{\mathfrak{s}}}\,\mathrm{Id}_{V_{\mathfrak{s}}}, \qquad \overleftarrow{\varphi}_{\mathfrak{s},\mathfrak{t}}\overleftarrow{\varphi}_{\mathfrak{s},\mathfrak{t}}^{\mathrm{T}} = \frac{\dim V_{\mathfrak{t}}}{\dim V_{\mathfrak{s}}}\,\mathrm{Id}_{V_{\mathfrak{s}}^*} \tag{4.14}$$

for any sectors $\mathfrak{s} \subset \mathfrak{t}$. As a result, we obtain

Lemma 4.3. *Let* $\mathfrak{s} \subset \mathfrak{t}$ *be any sectors.*

(1) $\overrightarrow{\varphi}_{\mathfrak{s},\mathfrak{t}} : V_{\mathfrak{t}} \to V_{\mathfrak{s}}$ *is surjective.*
(2) $\overleftarrow{\varphi}_{\mathfrak{s},\mathfrak{t}} : V^*_{\mathfrak{s}} \to V^*_{\mathfrak{t}}$ *is injective.*

The kernel of $\overrightarrow{\varphi}_{\mathfrak{s},\mathfrak{t}}$ and the cokernel of $\overleftarrow{\varphi}_{\mathfrak{s},\mathfrak{t}}$ will be the key in our derivation of the spectral duality in section 4.5.

By now it should be clear that $\overrightarrow{\varphi}_{\mathfrak{t}\backslash\{n\},\mathfrak{t}}$ kills ket vectors in a sector $\mathfrak{t}$ or maps them to the sector $\mathfrak{t}\backslash\{n\}$ in the Hasse diagram against one of the arrows. Similarly, $\overleftarrow{\varphi}_{\mathfrak{s},\mathfrak{s}\cup\{n\}}$ never kills bra vectors in a sector $\mathfrak{s}$ and maps them to the sector $\mathfrak{s} \cup \{n\}$ in the Hasse diagram along one of the arrows. Up to an overall scalar, $\overrightarrow{\varphi}_{\mathfrak{s},\mathfrak{t}}$ and $\overrightarrow{\varphi}^{\mathrm{T}}_{\mathfrak{s},\mathfrak{t}}$ coincide with composite actions of the $sl(N)$ generators e_i and f_i, respectively.

4.3. Commutativity of $\varphi_{\mathfrak{s},\mathfrak{t}}$ and the Hamiltonian

The action $H_{\mathfrak{s}} : V_{\mathfrak{s}} \to V_{\mathfrak{s}}$ of our Hamiltonian (4.8) is specified by (2.6) and (2.7) as

$$H_{\mathfrak{s}}|k_1,\ldots,k_L\rangle = \sum_{i\in\mathbb{Z}_L} \Theta(k_i - k_{i+1})(|k_1\ldots k_{i+1},k_i\ldots k_L\rangle - |k_1\ldots k_i,k_{i+1}\ldots k_L\rangle). \qquad (4.15)$$

Proposition 4.4. $\varphi_{\mathfrak{s},\mathfrak{t}}$ *is spectrum preserving. Namely,* $\varphi_{\mathfrak{s},\mathfrak{t}}H_{\mathfrak{t}} = H_{\mathfrak{s}}\varphi_{\mathfrak{s},\mathfrak{t}}$ *holds for any sectors* $\mathfrak{s} \subset \mathfrak{t}$.

Proof. Consider the actions on the ket vector $|k\rangle = |k_1,\ldots,k_L\rangle \in V_{\mathfrak{t}}$:

$$\varphi_{\mathfrak{s},\mathfrak{t}}H_{\mathfrak{t}}|k\rangle = \sum_{i\in\mathbb{Z}_L} \Theta(k_i - k_{i+1})(|k'_1\ldots k'_{i+1},k'_i\ldots k'_L\rangle - |k'_1\ldots k'_i,k'_{i+1}\ldots k'_L\rangle), \qquad (4.16)$$

$$H_{\mathfrak{s}}\varphi_{\mathfrak{s},\mathfrak{t}}|k\rangle = \sum_{i\in\mathbb{Z}_L} \Theta(k'_i - k'_{i+1})(|k'_1\ldots k'_{i+1},k'_i\ldots k'_L\rangle - |k'_1\ldots k'_i,k'_{i+1}\ldots k'_L\rangle), \qquad (4.17)$$

where x' is the one specified in (4.12). For simplicity, let us write (k_i, k_{i+1}) as (x, y). From (4.12), we see that $x > y$ implies $x' \geqslant y'$, and similarly $x < y$ implies $x' \leqslant y'$. From this fact and the definition of Θ in (2.3), the discrepancy of the coefficients $\Theta(x - y)$ and $\Theta(x' - y')$ in the above two formulae can possibly make difference only when ($x > y$ and $x' = y'$) or ($x < y$ and $x' = y'$). But in both cases, the vector $|\ldots y', x'\ldots\rangle - |\ldots x', y'\ldots\rangle$ is zero. Thus the right-hand sides of (4.16) and (4.17) are the same. □

Our Hamiltonian arises as an expansion coefficient of a commuting transfer matrix $T(\lambda)$ with respect to the spectral parameter λ (see (5.8)). However, the commutativity $\varphi_{\mathfrak{s},\mathfrak{t}}T(\lambda)_{\mathfrak{t}} = T(\lambda)_{\mathfrak{s}}\varphi_{\mathfrak{s},\mathfrak{t}}$ does not hold in general.

To each sector $\mathfrak{s} = \{s_1 < \cdots < s_{n-1}\} \in \mathcal{S}$, we associate

$$\mathrm{Spec}(\mathfrak{s}) = \text{multiset of eigenvalues of } H_{\mathfrak{s}}, \qquad (4.18)$$

where the multiplicity of an element represents, of course, the degree of its degeneracy. This definition is just a translation of (3.4) into the notation (4.8). The property (3.5) reads

$$\mathrm{Spec}(s_1,\ldots,s_{n-1}) = \mathrm{Spec}(L - s_{n-1},\ldots,L - s_1). \qquad (4.19)$$

One has $\sharp\mathrm{Spec}(\mathfrak{s}) = \dim V_{\mathfrak{s}} = \dim V^*_{\mathfrak{s}}$. Lemma 4.3 (2) and proposition 4.4 lead to the following theorem.

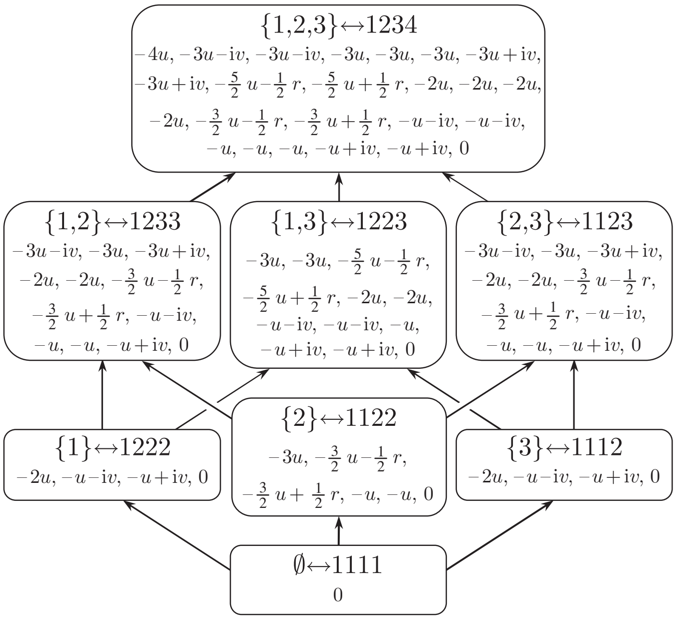

Figure 5. Spec($\mathfrak{s}$) for $L = 4$. $u = p + q, v = p - q, r = \sqrt{-7p^2 + 18pq - 7q^2}$. The symmetry (4.19) can be also observed.

Theorem 4.5. *There is an embedding of the spectrum* Spec($\mathfrak{s}$) $\hookrightarrow$ Spec($\mathfrak{t}$) *for any pair of sectors such that* $\mathfrak{s} \subset \mathfrak{t}$*. In particular,* Spec($\Omega$) *contains the eigenvalues of the Hamiltonian* $H_{\mathfrak{s}}$ *of all the sectors* $\mathfrak{s} \in \mathcal{S}$*.*

See figure 5 for example.

4.4. Spectral duality in the maximal sector Ω

As indicated in theorem 4.5, the structure of the spectrum in the maximal sector $\Omega \in \mathcal{S}$ is of basic importance. In this subsection we concentrate on this sector and elucidate a duality.

Define a $\mathbb{C}$-linear map ω by

$$\begin{array}{cccc} \omega: & V_\Omega^* & \stackrel{\sim}{\longrightarrow} & V_\Omega \\ & \langle k_1, \ldots, k_L| & \mapsto & \operatorname{sgn}(k)|k_L, \ldots, k_1\rangle, \end{array} \tag{4.20}$$

where $\operatorname{sgn}(k) = \operatorname{sgn}(k_1, \ldots, k_L)$ stands for the signature of the permutation. (Note that $\mathcal{P}(\Omega)$ is the set of permutations of $(1, 2, \ldots, L)$.) Obviously, ω is bijective.

It turns out that ω interchanges the eigenvalues of the Hamiltonian as $E \leftrightarrow -L(p+q)-E$.

Theorem 4.6. *Let* $\langle\phi| \in V_\Omega^*$ *be an eigenvector such that* $\langle\phi|H_\Omega = E\langle\phi|$*. Set* $|\psi\rangle = \omega(\langle\phi|) \in V_\Omega$*. Then* $H_\Omega|\psi\rangle = (-L(p+q) - E)|\psi\rangle$ *holds.*

Proof. Let $\langle\phi| = \sum_{k\in\mathcal{P}(\Omega)} f(k_1, \ldots, k_L)\langle k_1, \ldots, k_L|$. Then $\langle\phi|H_\Omega = E\langle\phi|$ is expressed as

$$\sum_{i\in\mathbb{Z}_L} \Theta(k_i - k_{i+1})(f(k^{(i)}) - f(k)) = Ef(k),$$

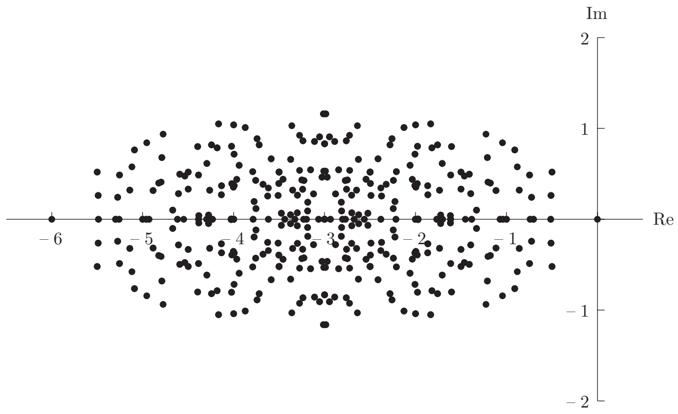

Figure 6. Spec(Ω) for $L = 6$ and $(p, q) = (0.8, 0.2)$. The symmetry with respect to $-L(p+q)/2 = -3$ can be observed.

where we have used the shorthand $k = (k_1, \ldots, k_i, k_{i+1}, \ldots, k_L)$ and $k^{(i)} = (k_1, \ldots, k_{i+1}, k_i, \ldots, k_L)$. Adding $(p+q)Lf(k)$ on both sides we obtain

$$\sum_{i\in\mathbb{Z}_L} \Theta(k_i - k_{i+1}) f(k^{(i)}) + \sum_{i\in\mathbb{Z}_L} (p + q - \Theta(k_i - k_{i+1})) f(k) = (E + L(p+q)) f(k).$$

Since k_i's are all distinct in the sector Ω under consideration, the coefficient in the second term equals $\Theta(k_{i+1} - k_i)$. Multiplication of $-\text{sgn}(k) = \text{sgn}(k^{(i)})$ on both sides leads to

$$\sum_{i\in\mathbb{Z}_L} \Theta(k_i - k_{i+1}) \text{sgn}(k^{(i)}) f(k^{(i)}) - \sum_{i\in\mathbb{Z}_L} \Theta(k_{i+1} - k_i) \text{sgn}(k) f(k)$$
$$= (-E - L(p+q)) \text{sgn}(k) f(k).$$

This coincides with the equation $H_\Omega|\psi\rangle = (-L(p+q) - E)|\psi\rangle$ with $|\psi\rangle = \sum_{k\in\mathcal{P}(\Omega)} \text{sgn}(k) f(k_1, \ldots, k_L)|k_L, \ldots, k_1\rangle$. □

Remark 4.7. It is easy to see that $\langle\phi| = \sum_{k\in\mathcal{P}(\Omega)} \langle k_1, \ldots, k_L| \in V_\Omega^*$ is the eigen-bra vector with the largest eigenvalue $E = 0$. It follows that $\omega(\langle\phi|) \in V_\Omega$ is the eigen-ket vector with the smallest eigenvalue $-L(p+q)$. Namely, one has

$$(H_\Omega + L(p+q)) \sum_{k\in\mathcal{P}(\Omega)} \text{sgn}(k)|k_L, \ldots, k_1\rangle = 0. \tag{4.21}$$

In view of conjecture 5.1 and the remark following it, we assume the diagonalizability of the Hamiltonian H_Ω.[5] Then every eigenvalue in Spec(Ω) is associated with an eigenvector in V_Ω^*. Therefore theorem 4.6 implies the following corollary.

Corollary 4.8. Spec$(\Omega) = -L(p+q) -$ Spec(Ω).

Figure 6 is a plot showing this property. The property of interchanging the eigenvalues of the Hamiltonian $E \leftrightarrow -L(p+q) - E$ will be referred as *spectrum reversing*. Our main task in the following is to extend ω to a spectrum reversing operator between general sectors, and to identify the 'genuine components' that are in bijective correspondence thereunder. This will be achieved as ω° in theorem 4.12.

[5] Theorem 4.5 is derived on the basis of generalized eigenvectors and hence its validity is independent of the diagonalizability of the Hamiltonian.

4.5. Genuine components $X^*_{\mathfrak{s}}$ and $Y_{\mathfrak{s}}$

Theorem 4.5 motivates us to classify the eigenvalues $\mathrm{Spec}(\mathfrak{s})$ in a sector $\mathfrak{s}$ into two kinds. One is those coming from the smaller sectors $\mathfrak{u} \subset \mathfrak{s}$ through the embedding $\mathrm{Spec}(\mathfrak{u}) \hookrightarrow \mathrm{Spec}(\mathfrak{s})$. The other is the *genuine eigenvalues* that are born at $\mathfrak{s}$ without such an origin. Having this feature in mind we introduce a quotient $X^*_{\mathfrak{s}}$ of $V^*_{\mathfrak{s}}$ and a subspace $Y_{\mathfrak{s}}$ of $V_{\mathfrak{s}}$ as

$$X^*_{\mathfrak{s}} = V^*_{\mathfrak{s}} \Big/ \sum_{\mathfrak{u}\subset\mathfrak{s}} \mathrm{Im}\, \overleftarrow{\varphi}_{\mathfrak{u},\mathfrak{s}}, \qquad Y_{\mathfrak{s}} = \bigcap_{\mathfrak{u}\subset\mathfrak{s}} \mathrm{Ker}\, \overrightarrow{\varphi}_{\mathfrak{u},\mathfrak{s}}. \tag{4.22}$$

We call $X^*_{\mathfrak{s}}$ and $Y_{\mathfrak{s}}$ the *genuine component* of $V^*_{\mathfrak{s}}$ and $V_{\mathfrak{s}}$, respectively. (We set $X^*_{\emptyset} = V^*_{\emptyset} = \mathbb{C}\langle 1,\ldots,1|$ and $Y_{\emptyset} = V_{\emptyset} = \mathbb{C}|1,\ldots,1\rangle$.) The Hamiltonian $H_{\mathfrak{s}}$ acts on each $X^*_{\mathfrak{s}}$ and $Y_{\mathfrak{s}}$ owing to proposition 4.4. The vector spaces $X^*_{\mathfrak{s}}$ and $Y_{\mathfrak{s}}$ are dual to each other canonically, therefore

$$\dim X^*_{\mathfrak{s}} = \dim Y_{\mathfrak{s}}. \tag{4.23}$$

We wish to focus on the spectra that are left after excluding the embedding structure explained above and in theorem 4.5. This leads us to define the set of genuine eigenvalues of a sector $\mathfrak{s}$ as

$$\begin{aligned}\mathrm{Spec}^{\circ}(\mathfrak{s}) &= \text{multiset of eigenvalues of } H_{\mathfrak{s}}|_{X^*_{\mathfrak{s}}}\\ &= \text{multiset of eigenvalues of } H_{\mathfrak{s}}|_{Y_{\mathfrak{s}}}.\end{aligned} \tag{4.24}$$

Let us write the image of $\langle\phi| \in V^*_{\mathfrak{s}}$ in $X^*_{\mathfrak{s}}$ under the natural projection by $[\langle\phi|]$. Fix an embedding of $X^*_{\mathfrak{s}}$ into $V^*_{\mathfrak{s}}$ sending each eigenvector $[\langle\phi|] \in X^*_{\mathfrak{s}}$ to an eigenvector $\langle\phi'| \in V^*_{\mathfrak{s}}$ with the same eigenvalue satisfying $[\langle\phi|] = [\langle\phi'|]$. The image of the embedding is complementary to $\sum_{\mathfrak{u}\subset\mathfrak{s}} \mathrm{Im}\, \overleftarrow{\varphi}_{\mathfrak{u},\mathfrak{s}}$, therefore we can treat the first relation in (4.22) as $V^*_{\mathfrak{s}} = X^*_{\mathfrak{s}} \oplus \sum_{\mathfrak{u}\subset\mathfrak{s}} \mathrm{Im}\, \overleftarrow{\varphi}_{\mathfrak{u},\mathfrak{s}}$. Then the following decomposition holds:

$$V^*_{\mathfrak{s}} = \bigoplus_{\mathfrak{u}\subseteq\mathfrak{s}} X^*_{\mathfrak{u}} \overleftarrow{\varphi}_{\mathfrak{u},\mathfrak{s}}. \tag{4.25}$$

From theorem 4.5 and (4.25) we have

$$\mathrm{Spec}(\mathfrak{s}) = \bigcup_{\mathfrak{u}\subseteq\mathfrak{s}} \mathrm{Spec}^{\circ}(\mathfrak{u}), \tag{4.26}$$

where the multiplicity is taken into account for the union of the multisets. In terms of the cardinality, this amounts to

$$\dim V^*_{\mathfrak{s}} = \sum_{\mathfrak{u}\subseteq\mathfrak{s}} \dim X^*_{\mathfrak{u}}. \tag{4.27}$$

Theorem 4.9 (dimensional duality). *For any sector $\mathfrak{s} \in \mathcal{S}$, the following equality is valid:*

$$\dim X^*_{\mathfrak{s}} = \dim X^*_{\bar{\mathfrak{s}}},$$

or equivalently $\sharp\mathrm{Spec}^{\circ}(\mathfrak{s}) = \sharp\mathrm{Spec}^{\circ}(\bar{\mathfrak{s}})$. Here $\bar{\mathfrak{s}}$ denotes the complement sector (4.4).

See figure 7 for example with $L = 4$. The proof is due to the standard Möbius inversion in the poset $\mathcal{S}$ and available in appendix A.

The following lemma, although slightly technical, plays a key role in our subsequent argument.

Lemma 4.10.

(1) $\overrightarrow{\varphi}_{\Omega\setminus\{r\},\Omega}\,\omega(\mathrm{Im}\overleftarrow{\varphi}_{\Omega\setminus\{r\},\Omega}) = 0$ *for any* $r \in \Omega$.

(2) $\overrightarrow{\varphi}_{\mathfrak{s},\Omega}\,\omega(\mathrm{Im}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}) = 0$ *unless* $\mathfrak{u} \supseteq \bar{\mathfrak{s}}$.

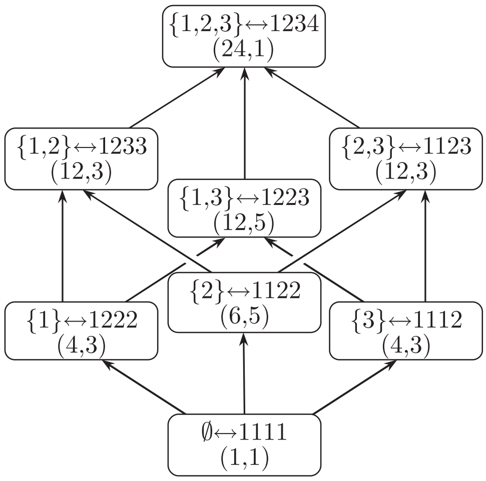

Figure 7. The data $(\dim V^*_{\mathfrak{s}}, \dim X^*_{\mathfrak{s}}) = (\sharp\mathrm{Spec}(\mathfrak{s}), \sharp\mathrm{Spec}^\circ(\mathfrak{s}))$ are presented for each $\mathfrak{s}$ in the same Hasse diagram (*c*) in figure 4. The dimensional duality (theorem 4.9) can be observed. For a systematic calculation of these data, see appendix A.

Proof. (1) For brevity we write $\Omega_r = \Omega\backslash\{r\}$. We illustrate an example $L = 5, \Omega = \{1, 2, 3, 4\}, \Omega_2 = \{1, 3, 4\}$, from which the general case is easily understood. Recall the scheme as in (4.11):

$$\mathcal{P}(\Omega = \{1,2,3,4\}): \quad 1\overset{1}{|}2\overset{2}{|}3\overset{3}{|}4\overset{4}{|}5,$$

$$\mathcal{P}(\Omega_2 = \{1,3,4\}): \quad 1\overset{1}{|}22\overset{3}{|}3\overset{4}{|}4.$$

Thus $\overleftarrow{\varphi}_{\Omega_2,\Omega}$ is the operator replacing the local states $3 \to 4, 4 \to 5$ and moreover changes $\langle \ldots 2, \ldots, 2, \ldots|$ into the symmetric sum $\langle \ldots 3, \ldots, 2, \ldots| + \langle \ldots 2, \ldots, 3, \ldots|$. At the next stage, ω in (4.20) attaches the factor $\mathrm{sgn}(k)$ which makes the above sum antisymmetric. Finally, $\overrightarrow{\varphi}_{\Omega_2,\Omega}$ makes the antisymmetrized letters 2 and 3 merge into 2 again (and also does $4 \to 3, 5 \to 4$), which therefore kills the vector. For example,

$$\begin{aligned}\langle 42312| &\overset{\overleftarrow{\varphi}_{\Omega_2,\Omega}}{\longmapsto} \langle 52413| + \langle 53412| \\ &\overset{\omega}{\longmapsto} -|31425\rangle + |21435\rangle \\ &\overset{\overrightarrow{\varphi}_{\Omega_2,\Omega}}{\longmapsto} -|21324\rangle + |21324\rangle = 0.\end{aligned}$$

(2) Note that $\mathrm{Im}\overleftarrow{\varphi}_{\mathfrak{u},\Omega} = V^*_{\mathfrak{u}}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}$. Thus we are to ask when $\overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(V^*_{\mathfrak{u}}\overleftarrow{\varphi}_{\mathfrak{u},\Omega})$ vanishes. It is helpful to view this as a process in the Hasse diagram going from $V^*_{\mathfrak{u}}$ to $V_{\mathfrak{s}}$ via the maximal sector Ω as in figure 8, where $\overline{\mathfrak{u}} = \{\overline{u}_1, \ldots, \overline{u}_a\}$ and $\overline{\mathfrak{s}} = \{\overline{s}_1, \ldots, \overline{s}_b\}$. In figure 8, the arrows $\nearrow$ represent the factorization $\overleftarrow{\varphi}_{\mathfrak{u},\Omega} = \overleftarrow{\varphi}_{\mathfrak{u},\mathfrak{u}\cup\{\overline{u}_a\}} \cdots \overleftarrow{\varphi}_{\Omega\backslash\{\overline{u}_1\},\Omega}$ due to lemma 4.2 growing $\mathfrak{u}$ up to Ω by adding $\overline{u}_i$'s one by one. Similarly the arrows $\searrow$ stand for $\overrightarrow{\varphi}_{\mathfrak{s},\Omega} = \overrightarrow{\varphi}_{\mathfrak{s},\mathfrak{s}\cup\{\overline{s}_b\}} \cdots \overrightarrow{\varphi}_{\Omega\backslash\{\overline{s}_1\},\Omega}$ shrinking Ω down to $\mathfrak{s}$ by removing $\overline{s}_i$'s one by one. (The arrows attached to $\overline{u}_i$ ($\overline{s}_i$) are the same (opposite) as those in the Hasse diagram.) In this way

$$\begin{aligned}\overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(V^*_{\mathfrak{u}}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}) &= \cdots \overrightarrow{\varphi}_{\Omega\backslash\{\overline{s}_1\},\Omega}\omega(\cdots \overleftarrow{\varphi}_{\Omega\backslash\{\overline{u}_1\},\Omega}) \\ &= \cdots \overrightarrow{\varphi}_{\Omega\backslash\{\overline{s}_i\},\Omega}\omega(\cdots \overleftarrow{\varphi}_{\Omega\backslash\{\overline{u}_j\},\Omega}) \quad \text{for any } 1 \leqslant i \leqslant b, 1 \leqslant j \leqslant a,\end{aligned}$$

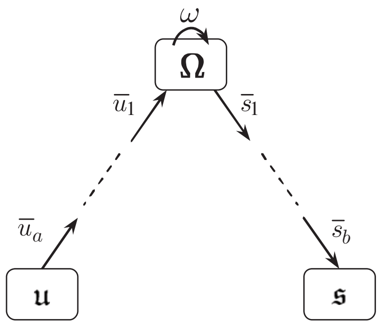

Figure 8. A conceptual scheme of the proof of lemma 4.10 (2).

where the second equality is due to lemma 4.2 which assures that the factorization is possible in arbitrary orders. From the assertion (1) we thus find that this vanishes if $\overline{\mathfrak{s}} \cap \overline{\mathfrak{u}} \neq \emptyset$. In other words, $\overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(V_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}) = 0$ unless $\emptyset = \overline{\mathfrak{s}} \cap \overline{\mathfrak{u}}$, or equivalently $\mathfrak{u} \supseteq \overline{\mathfrak{s}}$. □

Proposition 4.11.

(1) $V_{\mathfrak{s}} = \bigoplus_{\mathfrak{u}\supseteq\overline{\mathfrak{s}}} \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega})$.

(2) $Y_{\mathfrak{s}} = \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\overline{\mathfrak{s}}}^{*}\overleftarrow{\varphi}_{\overline{\mathfrak{s}},\Omega})$.

Proof. (1)

$$V_{\mathfrak{s}} \overset{\text{Lem. 4.3(1)}}{=} \overrightarrow{\varphi}_{\mathfrak{s},\Omega}V_{\Omega} = \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(V_{\Omega}^{*}) \overset{(4.25)}{=} \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega\left(\bigoplus_{\mathfrak{u}} X_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}\right)$$
$$= \sum_{\mathfrak{u}} \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}) \overset{\text{Lem. 4.10(2)}}{=} \sum_{\mathfrak{u}\supseteq\overline{\mathfrak{s}}} \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}). \tag{4.28}$$

Taking the dimensions, we have

$$\dim V_{\mathfrak{s}} = \dim \sum_{\mathfrak{u}\supseteq\overline{\mathfrak{s}}} \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega}) \leqslant \sum_{\mathfrak{u}\supseteq\overline{\mathfrak{s}}} \dim \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\mathfrak{u}}^{*}\overleftarrow{\varphi}_{\mathfrak{u},\Omega})$$
$$\leqslant \sum_{\mathfrak{u}\supseteq\overline{\mathfrak{s}}} \dim X_{\mathfrak{u}}^{*} \overset{\text{Th. 4.9}}{=} \sum_{\mathfrak{u}\supseteq\overline{\mathfrak{s}}} \dim X_{\overline{\mathfrak{u}}}^{*} = \sum_{\overline{\mathfrak{u}}\subseteq\mathfrak{s}} \dim X_{\overline{\mathfrak{u}}}^{*} \overset{(4.27)}{=} \dim V_{\mathfrak{s}}^{*} = \dim V_{\mathfrak{s}}.$$

Thus all the inequalities $\leqslant$ here are actually the equality $=$. Moreover, all the sums $\sum$ in (4.28) must be the direct sum $\oplus$, completing the proof.

(2) Let $\tilde{Y}_{\mathfrak{s}} = \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\overline{\mathfrak{s}}}^{*}\overleftarrow{\varphi}_{\overline{\mathfrak{s}},\Omega})$. By an argument similar to the proof of lemma 4.10 (2), one can easily show that $\tilde{Y}_{\mathfrak{s}}$ is killed by $\overrightarrow{\varphi}_{\mathfrak{s}\setminus\{n\},\mathfrak{s}}$ for any $n \in \mathfrak{s}$. In view of lemma 4.2, this implies $\tilde{Y}_{\mathfrak{s}} \subseteq Y_{\mathfrak{s}}$. The proof is completed by noting $\dim \tilde{Y}_{\mathfrak{s}} = \dim \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\omega(X_{\overline{\mathfrak{s}}}^{*}\overleftarrow{\varphi}_{\overline{\mathfrak{s}},\Omega}) \overset{(1)}{=} \dim X_{\overline{\mathfrak{s}}}^{*} \overset{\text{Th. 4.9}}{=} \dim X_{\mathfrak{s}}^{*} \overset{(4.23)}{=} \dim Y_{\mathfrak{s}}$. □

Combining proposition 4.11 (2) and theorem 4.6, we arrive at our main result in this section.

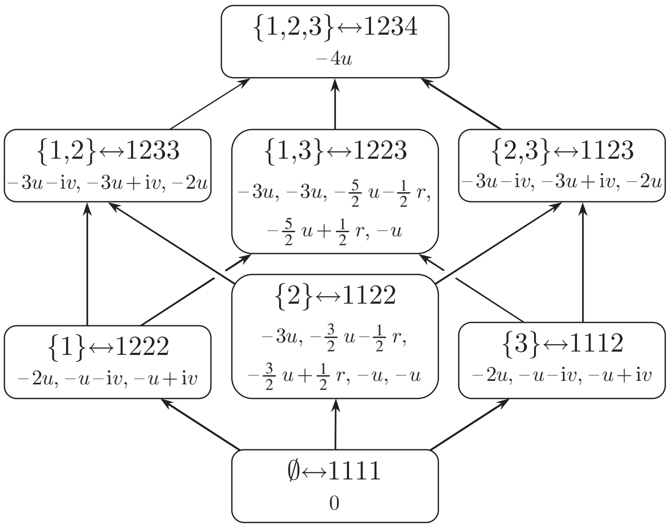

Figure 9. $\mathrm{Spec}^{\circ}(\mathfrak{s})$ for $L = 4$. $u = p + q$, $v = p - q$, $r = \sqrt{-7p^2 + 18pq - 7q^2}$.

Theorem 4.12 (spectral duality). *For any sector $\mathfrak{s} \in \mathcal{S}$ and its complementary sector $\overline{\mathfrak{s}}$, there is a spectrum reversing bijection ω° between their genuine components:*

$$\begin{array}{llll} \omega^{\circ}: & X^{*}_{\overline{\mathfrak{s}}} & \stackrel{\sim}{\longrightarrow} & Y_{\mathfrak{s}} \\ & \langle\phi| & \mapsto & \overrightarrow{\varphi}_{\mathfrak{s},\Omega}\,\omega(\langle\phi|\overleftarrow{\varphi}_{\overline{\mathfrak{s}},\Omega}). \end{array} \tag{4.29}$$

In particular, the genuine spectrum enjoys the following duality:

$$\mathrm{Spec}^{\circ}(\overline{\mathfrak{s}}) = -L(p+q) - \mathrm{Spec}^{\circ}(\mathfrak{s}). \tag{4.30}$$

This relation is a refinement of theorem 4.9.

Example 4.13. Figure 9 presents $\mathrm{Spec}^{\circ}(\mathfrak{s})$ for $L = 4$ in the same format as figure 5. All the genuine eigenvalues form pairs with those in the complementary sectors to add up to $-L(p+q) = -4u$ including the multiplicity. The full spectrum $\mathrm{Spec}(\mathfrak{s})$ in figure 5 is reproduced from the data in figure 9 and (4.26).

Remark 4.14. The genuine spectrum Spec° also enjoys the symmetry (4.19). It follows that if a sector $\mathfrak{t}$ satisfies $\mathfrak{t} \supset \mathfrak{s}, \tilde{\mathfrak{s}}$ with $\mathfrak{s} = (s_1, \ldots, s_{n-1})$ and $\tilde{\mathfrak{s}} = (L - s_{n-1}, \ldots, L - s_1)$, then $H_{\mathfrak{t}}$ is degenerated because of $\mathrm{Spec}(\mathfrak{t}) \supset \mathrm{Spec}^{\circ}(\mathfrak{s}) \cup \mathrm{Spec}^{\circ}(\tilde{\mathfrak{s}})$ and $\mathrm{Spec}^{\circ}(\mathfrak{s}) = \mathrm{Spec}^{\circ}(\tilde{\mathfrak{s}})$.

5. Integrability of the model

Our multi-species ASEP is integrable in the sense that the eigenvalue formula of the Hamiltonian can be derived by a nested Bethe ansatz [Sc]. See also [AB, BDV].

As mentioned in section 1, our Hamiltonian is associated with the transfer matrix of the Perk–Schultz vertex model [PS]. In section 5.1, we derive the eigenvalues of the transfer matrix in a slightly more general way than [Sc]. Namely we execute the nested Bethe ansatz in an arbitrary 'nesting order'. In section 5.2, we utilize it to give an alternative account of the spectral inclusion property (theorem 4.5) in the Bethe ansatz framework. We also recall the original derivation of the asymptotic form of the spectrum following [K]. In section 5.3, the Bethe ansatz results are presented in a more conventional parameterization with the spectral parameter having a difference property.

5.1. Nested algebraic Bethe ansatz

5.1.1. Transfer matrix and eigenvalue formula. Let us derive the eigenvalues of the Hamiltonian H (2.6) for the $(N-1)$-species ASEP on the ring $\mathbb{Z}_L$ by using the nested algebraic Bethe ansatz. Let W_j be a vector space $W = \mathbb{C}^N$ at the jth site of the ring. We define a matrix $R_{jk}(\lambda) \in \mathrm{End}(W_j \otimes W_k)$ as

$$R_{jk}(\lambda) = P_{jk}(1 + \lambda h_{jk}), \tag{5.1}$$

where P_{jk} and h_{jk} are, respectively, the permutation operator and the local Hamiltonian (2.7) acting non-trivially on $W_j \otimes W_k$. The non-zero elements are explicitly given by

$$R^{\alpha\alpha}_{\alpha\alpha}(\lambda) = 1, \qquad R^{\alpha\beta}_{\alpha\beta}(\lambda) = \begin{cases} q\lambda & \text{for } \alpha < \beta, \\ p\lambda & \text{for } \alpha > \beta, \end{cases} \quad R^{\beta\alpha}_{\alpha\beta}(\lambda) = \begin{cases} 1 - q\lambda & \text{for } \alpha < \beta, \\ 1 - p\lambda & \text{for } \alpha > \beta. \end{cases} \tag{5.2}$$

Here $\alpha, \beta \in \{1, 2, \ldots, N\}$, and $R^{\gamma\delta}_{\alpha\beta}(\lambda)$ stands for $R_{jk}(\lambda)(|\alpha\rangle_j \otimes |\beta\rangle_k) = |\gamma\rangle_j \otimes |\delta\rangle_k R^{\gamma\delta}_{\alpha\beta}(\lambda)$ (summation over repeated indices will always be assumed). The above R-matrix satisfies the Yang–Baxter equation [Ba]

$$R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda) = R_{12}(\lambda)R_{13}(\lambda_1)R_{23}(\lambda_2), \tag{5.3}$$

where the parameter λ is given by

$$\lambda = \xi(\lambda_1, \lambda_2) = \frac{\lambda_1 - \lambda_2}{1 - (p+q)\lambda_2 + pq\lambda_1\lambda_2}. \tag{5.4}$$

This is not a simple difference $\lambda_1 - \lambda_2$. However, one can restore the difference property by changing variables as in section 5.3. Thanks to (5.3), the transfer matrix $T(\lambda) \in \mathrm{End}(W^{\otimes L})$

$$T(\lambda) = \mathrm{tr}_{W_0}[R_{0L}(\lambda) \cdots R_{01}(\lambda)] \tag{5.5}$$

constitutes a one-parameter commuting family

$$[T(\lambda_1), T(\lambda_2)] = 0. \tag{5.6}$$

It means that $T(\lambda)$ is a generating function for a set of mutually commuting 'quantum integrals of motion' $\mathcal{I}_j$ $(j = 1, 2, \ldots)$:

$$\mathcal{I}_j = \left(\frac{\partial}{\partial\lambda}\right)^j \ln T(\lambda)\bigg|_{\lambda=0}. \tag{5.7}$$

$\mathcal{I}_0$ is the momentum operator related to the shift operator C (2.9) by $C = \exp \mathcal{I}_0$. $\mathcal{I}_1$ yields the ASEP Hamiltonian H (2.6):

$$\mathcal{I}_1 = \sum_{j\in\mathbb{Z}_L} R_{jj+1}(0)R'_{jj+1}(0) = \sum_{j\in\mathbb{Z}_L} h_{jj+1} = H. \tag{5.8}$$

Thus the eigenvalue problem of H is contained in that of $T(\lambda)$. To find the eigenvalues of $T(\lambda)$, we introduce the monodromy matrix $\mathcal{T}(\lambda) \in \mathrm{End}(W_0 \otimes W^{\otimes L})$ by

$$\mathcal{T}(\lambda) = R_{0L}(\lambda) \cdots R_{01}(\lambda). \tag{5.9}$$

Its trace over the auxiliary space W_0 reproduces the transfer matrix (5.5)

$$T(\lambda) = \mathrm{tr}_{W_0} \mathcal{T}(\lambda). \tag{5.10}$$

From the Yang–Baxter equation (5.3), one sees the following is valid:

$$\mathcal{T}_2(\lambda_2)\mathcal{T}_1(\lambda_1)R_{12}(\lambda) = R_{12}(\lambda)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2), \tag{5.11}$$

where $R_{12}(\lambda)$ here acts on the tensor product of two auxiliary spaces.

Let us define the elements of the monodromy matrix in the auxiliary space as $\mathcal{T}(\lambda)|\alpha\rangle_0 = \mathcal{T}_\alpha^\beta(\lambda)|\beta\rangle_0$, where $\mathcal{T}_\alpha^\beta(\lambda)$ acts on the quantum space $W^{\otimes L}$. More explicitly,

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{T}_{a_1}^{a_1}(\lambda) & B_{a_2}(\lambda) & \cdots & B_{a_N}(\lambda) \\ C^{a_2}(\lambda) & \mathcal{T}_{a_2}^{a_2}(\lambda) & \cdots & \mathcal{T}_{a_N}^{a_2}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ C^{a_N}(\lambda) & \mathcal{T}_{a_2}^{a_N}(\lambda) & \cdots & \mathcal{T}_{a_N}^{a_N}(\lambda) \end{pmatrix}, \tag{5.12}$$
$$B_{a_j}(\lambda) := \mathcal{T}_{a_j}^{a_1}(\lambda), \quad C^{a_j}(\lambda) := \mathcal{T}_{a_1}^{a_j}(\lambda) \qquad \text{for} \quad j \in \{2, \ldots, N\}.$$

Here we have introduced the indices $a_1, \ldots, a_N$ that are arbitrary as long as $\{a_j\}_{j=1}^N = \{1, \ldots, N\}$. They specify the nesting order $|a_1\rangle, |a_2\rangle, \ldots, |a_N\rangle$.[6]

Let $|\text{vac}\rangle := |a_1\rangle_1 \otimes |a_1\rangle_2 \otimes \cdots \otimes |a_1\rangle_L$ be the 'vacuum state' in the quantum space. It immediately follows that the action of $\mathcal{T}(\lambda)$ on $|\text{vac}\rangle$ is given by

$$\mathcal{T}(\lambda)|\text{vac}\rangle = \begin{pmatrix} 1 & B_{a_2}(\lambda) & \cdots & B_{a_N}(\lambda) \\ 0 & d(\lambda)(q/p)^{L\theta_{12}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(\lambda)(q/p)^{L\theta_{1N}} \end{pmatrix} |\text{vac}\rangle, \tag{5.13}$$

where

$$d(\lambda) := (p\lambda)^L, \qquad \theta_{ij} := \theta(a_i - a_j) := \begin{cases} 0 & \text{for} \quad a_i < a_j, \\ 1 & \text{for} \quad a_i > a_j. \end{cases} \tag{5.14}$$

Using the relation (5.11), we can verify the following commutation relations:

$$B_\alpha(\lambda)B_\beta(\lambda') = \begin{cases} B_\beta(\lambda')B_\alpha(\lambda) & \text{for } \alpha = \beta, \\ R_{\alpha\beta}^{\gamma\delta}(\xi(\lambda,\lambda'))B_\delta(\lambda')B_\gamma(\lambda) & \text{for } \alpha \neq \beta, \end{cases}$$
$$\mathcal{T}_{a_1}^{a_1}(\lambda)B_\alpha(\lambda') = \begin{cases} f(\lambda',\lambda)B_\alpha(\lambda')\mathcal{T}_{a_1}^{a_1}(\lambda) + g(\lambda,\lambda')B_\alpha(\lambda)\mathcal{T}_{a_1}^{a_1}(\lambda') & \text{for } a_1 < \alpha, \\ \overline{f}(\lambda',\lambda)B_\alpha(\lambda')\mathcal{T}_{a_1}^{a_1}(\lambda) + \overline{g}(\lambda,\lambda')B_\alpha(\lambda)\mathcal{T}_{a_1}^{a_1}(\lambda') & \text{for } a_1 > \alpha, \end{cases}$$
$$\mathcal{T}_\gamma^\beta(\lambda)B_\alpha(\lambda') = \begin{cases} f(\lambda,\lambda')R_{\gamma\alpha}^{\varepsilon\delta}(\xi(\lambda,\lambda'))B_\delta(\lambda')\mathcal{T}_\varepsilon^\beta(\lambda) - g(\lambda,\lambda')B_\gamma(\lambda)\mathcal{T}_\alpha^\beta(\lambda') & \text{for } a_1 < \beta, \\ \overline{f}(\lambda,\lambda')R_{\gamma\alpha}^{\varepsilon\delta}(\xi(\lambda,\lambda'))B_\delta(\lambda')\mathcal{T}_\varepsilon^\beta(\lambda) - \overline{g}(\lambda,\lambda')B_\gamma(\lambda)\mathcal{T}_\alpha^\beta(\lambda') & \text{for } a_1 > \beta. \end{cases} \tag{5.15}$$

Here $\alpha, \beta, \gamma, \delta \in \{a_j\}_{j=2}^N$, and the functions $f, g, \overline{f}$ and $\overline{g}$ are defined by

$$f(\lambda,\mu) = \frac{1}{p\xi(\lambda,\mu)} = \frac{1-(p+q)\mu + pq\lambda\mu}{p(\lambda-\mu)},$$
$$g(\lambda,\mu) = 1 - f(\mu,\lambda) = f(\lambda,\mu) - \frac{q}{p} = \frac{(1-q\lambda)(1-p\mu)}{p(\lambda-\mu)}, \tag{5.16}$$
$$\overline{f}(\lambda,\mu) = f(\lambda,\mu)|_{p\leftrightarrow q} = \frac{p}{q}f(\lambda,\mu), \quad \overline{g}(\lambda,\mu) = g(\lambda,\mu)|_{p\leftrightarrow q}.$$

Consider the following state with the number of particles of the a_jth kind being m_{a_j}:

$$|\{\lambda^{(1)}\}\rangle = F^{\alpha_1\cdots\alpha_{n_1}}B_{\alpha_1}\big(\lambda_1^{(1)}\big)\cdots B_{\alpha_{n_1}}\big(\lambda_{n_1}^{(1)}\big)|\text{vac}\rangle, \alpha_j \in \{a_2, \ldots, a_N\} \quad (1 \leqslant j \leqslant n_1), \tag{5.17}$$

[6] In the standard nested algebraic Bethe ansatz, the nesting order is chosen as $a_j = j$.

where

$$n_k := \sum_{j=k+1}^{N} m_{a_j} \quad (1 \leqslant k \leqslant N-1; n_0 = L). \tag{5.18}$$

The sum over repeated indices in (5.17) is restricted by the condition

$$\sharp\{j | 1 \leqslant j \leqslant n_1, \alpha_j = a_k\} = m_{a_k} \quad (2 \leqslant k \leqslant N). \tag{5.19}$$

Then the action of $T(\lambda)$ on $|\{\lambda^{(1)}\}\rangle$ is calculated by using the relations (5.15) and (5.13)[7]:

$$T(\lambda)|\{\lambda^{(1)}\}\rangle = \left[\mathcal{T}_{a_1}^{a_1}(\lambda) + \sum_{\alpha=2}^{N} \mathcal{T}_{a_\alpha}^{a_\alpha}(\lambda)\right]|\{\lambda^{(1)}\}\rangle = \left(\frac{p}{q}\right)^{\overline{n}_1} \prod_{j=1}^{n_1} f\left(\lambda_j^{(1)}, \lambda\right)|\{\lambda^{(1)}\}\rangle$$
$$+ F^{\alpha_1\cdots\alpha_{n_1}} T^{(1)\beta_1,\ldots,\beta_{n_1}}_{\alpha_1,\ldots,\alpha_{n_1}}(\lambda|\{\lambda^{(1)}\}) d(\lambda) \prod_{j=1}^{n_1} f\left(\lambda, \lambda_j^{(1)}\right) \prod_{j=1}^{n_1} B_{\beta_j}\left(\lambda_j^{(1)}\right)|\text{vac}\rangle + \text{u.t.}, \tag{5.20}$$

where the product $\prod_{j=1}^{n_1} B_{\beta_j}\left(\lambda_j^{(1)}\right)$ is ordered from left to right with increasing j; $\alpha_j, \beta_j \in \{a_2, \ldots, a_N\}$; $\overline{n}_k$ is an integer given by

$$\overline{n}_k := \sum_{j=k+1}^{N} m_{a_j}\theta_{kj} \quad (1 \leqslant k \leqslant N-1), \tag{5.21}$$

and $T^{(1)\beta_1,\ldots,\beta_{n_1}}_{\alpha_1,\ldots,\alpha_{n_1}}(\lambda|\{\lambda^{(1)}\})$ is a matrix element of $T^{(1)}(\lambda|\{\lambda^{(1)}\}) \in \text{End}((\mathbb{C}^{N-1})^{\otimes L})$ defined by

$$T^{(1)}(\lambda|\{\lambda^{(1)}\}) = \sum_{\alpha=2}^{N} \left(\frac{q}{p}\right)^{(L-n_1)\theta_{1\alpha}} \left[R_{0n_1}\left(\xi\left(\lambda, \lambda_{n_1}^{(1)}\right)\right) \cdots R_{01}\left(\xi\left(\lambda, \lambda_1^{(1)}\right)\right)\right]_{a_\alpha}^{a_\alpha}. \tag{5.22}$$

Note that the sum corresponds to the trace over the $(N-1)$-dimensional auxiliary space spanned by the basis vectors $|a_j\rangle_0$ $(2 \leqslant j \leqslant N)$, and the quantum space acted on by $T^{(1)}(\lambda|\{\lambda^{(1)}\})$ is spanned by the vector $\otimes_{j=1}^{n_1}|\alpha_j\rangle_j$ where $\alpha_j \in \{a_k\}_{k=2}^{N}$.

If we set $F^{\alpha_1\cdots\alpha_{n_1}}$ as the elements of the eigenstate for $T^{(1)}(\lambda|\{\lambda^{(1)}\})$ and choose the set of unknown numbers $\left\{\lambda_j^{(1)}\right\}_{j=1}^{n_1}$ so that the unwanted terms (u.t.) in (5.20) become zero (u.t. = 0), the eigenvalue, written $\Lambda(\lambda)$, of the transfer matrix $T(\lambda)$ is expressed as

$$\Lambda(\lambda) = \left(\frac{p}{q}\right)^{\overline{n}_1} \prod_{j=1}^{n_1} f\left(\lambda_j^{(1)}, \lambda\right) + \Lambda^{(1)}(\lambda|\{\lambda^{(1)}\}) d(\lambda) \prod_{j=1}^{n_1} f\left(\lambda, \lambda_j^{(1)}\right). \tag{5.23}$$

Here $\Lambda^{(1)}(\lambda|\{\lambda^{(1)}\})$ is the eigenvalue of $T^{(1)}(\lambda|\{\lambda^{(1)}\})$, which are determined below. Noting that

$$\xi(\xi(\lambda_1, \mu), \xi(\lambda_2, \mu)) = \xi(\lambda_1, \lambda_2), \tag{5.24}$$

and using the Yang–Baxter equation (5.3), one finds that the transfer matrix $T^{(1)}(\lambda|\{\lambda^{(1)}\})$ forms a commuting family

$$[T(\lambda_1|\{\lambda^{(1)}\}), T(\lambda_2|\{\lambda^{(1)}\})] = 0. \tag{5.25}$$

Hence a method similar to the above is also applicable to the eigenvalue problem of $T^{(1)}(\lambda|\{\lambda^{(1)}\})$. Namely constructing the state

$$|\{\lambda^{(2)}\}\rangle = F^{(1)\alpha_1\cdots\alpha_{n_2}} B_{\alpha_1}^{(1)}\left(\lambda_1^{(2)}\right) \cdots B_{\alpha_{n_2}}^{(1)}\left(\lambda_{n_2}^{(2)}\right)|\text{vac}^{(1)}\rangle, \qquad |\text{vac}^{(1)}\rangle := \bigotimes_{j=1}^{n_2} |a_2\rangle_j, \tag{5.26}$$

[7] A good exposition of the calculations can be found in [Sc].

where $\alpha_j \in \{3, \ldots, N\}$ and

$$B^{(1)}_{\alpha_j}(\lambda) = \left[R_{0n_1}\left(\xi\left(\lambda, \lambda^{(1)}_{n_1}\right)\right) \cdots R_{01}\left(\xi\left(\lambda, \lambda^{(1)}_1\right)\right)\right]^{a_2}_{\alpha_j}, \tag{5.27}$$

we obtain

$$\Lambda^{(1)}(\lambda|\{\lambda^{(1)}_j\}) = \left(\frac{q}{p}\right)^{(L-n_1)\theta_{12}} \left(\frac{p}{q}\right)^{\overline{n}_2} \prod_{j=1}^{n_2} f\left(\lambda^{(2)}_j, \lambda\right)$$
$$+ \Lambda^{(2)}(\lambda|\{\lambda^{(2)}\}) \prod_{j=1}^{n_1} \frac{1}{f\left(\lambda, \lambda^{(1)}_j\right)} \prod_{j=1}^{n_2} f\left(\lambda, \lambda^{(2)}_j\right). \tag{5.28}$$

Note that the coefficients $F^{(1)\alpha_1\cdots\alpha_{n_2}}$ in (5.26) and $\Lambda^{(2)}(\lambda|\{\lambda^{(2)}\})$ in the above are, respectively, the elements of the eigenstate and the eigenvalue for the transfer matrix

$$T^{(2)}(\lambda|\{\lambda^{(2)}\}) = \sum_{\alpha=3}^{N} \left(\frac{q}{p}\right)^{(L-n_1)\theta_{1\alpha}+(n_1-n_2)\theta_{2\alpha}} \left[R_{\alpha n_2}\left(\xi\left(\lambda, \lambda^{(2)}_{n_2}\right)\right) \cdots R_{\alpha 1}\left(\xi\left(\lambda, \lambda^{(2)}_1\right)\right)\right]^{a_\alpha}_{a_\alpha}. \tag{5.29}$$

The sum corresponds to the trace over the $(N-2)$-dimensional auxiliary space spanned by the basis vectors $|a_j\rangle_0$ $(3 \leqslant j \leqslant N)$, and the space acted on by $T^{(2)}(\lambda|\{\lambda^{(2)}\})$ is spanned by $\otimes^{n_2}_{j=1}|\alpha_j\rangle_j$ where $\alpha_j \in \{a_k\}^N_{k=3}$. Repeating this procedure, one obtains

$$\Lambda^{(l)}(\lambda|\{\lambda^{(l)}\}) = \left(\frac{q}{p}\right)^{\sum_{j=1}^{l}(n_{j-1}-n_j)\theta_{jl+1}-\overline{n}_{l+1}} \prod_{j=1}^{n_{l+1}} f\left(\lambda^{(l+1)}_j, \lambda\right)$$
$$+ \Lambda^{(l+1)}(\lambda|\{\lambda^{(l+1)}\}) \prod_{j=1}^{n_l} \frac{1}{f\left(\lambda, \lambda^{(l)}_j\right)} \prod_{j=1}^{n_{l+1}} f\left(\lambda, \lambda^{(l+1)}_j\right), \tag{5.30}$$

for $2 \leqslant l < N-2$, and

$$\Lambda^{(N-2)}(\lambda|\{\lambda^{(N-2)}\}) = \left(\frac{q}{p}\right)^{\sum_{j=1}^{N-2}(n_{j-1}-n_j)\theta_{jN-1}-\overline{n}_{N-1}} \prod_{j=1}^{n_{N-1}} f\left(\lambda^{(N-1)}_j, \lambda\right)$$
$$+ \left(\frac{q}{p}\right)^{\sum_{j=1}^{N-1}(n_{j-1}-n_j)\theta_{jN}} \prod_{j=1}^{n_{N-2}} \frac{1}{f\left(\lambda, \lambda^{(N-2)}_j\right)} \prod_{j=1}^{n_{N-1}} f\left(\lambda, \lambda^{(N-1)}_j\right). \tag{5.31}$$

Thus we finally arrive at the eigenvalue formula of the transfer matrix:

$$\Lambda(\lambda) = \left(\frac{q}{p}\right)^{-\overline{n}_1} \prod_{j=1}^{n_1} f\left(\lambda^{(1)}_j, \lambda\right)$$
$$+ d(\lambda) \sum_{k=1}^{N-2} \left(\frac{q}{p}\right)^{\sum_{j=1}^{k}(n_{j-1}-n_j)\theta_{jk+1}-\overline{n}_{k+1}} \prod_{j=1}^{n_k} f\left(\lambda, \lambda^{(k)}_j\right) \prod_{j=1}^{n_{k+1}} f\left(\lambda^{(k+1)}_j, \lambda\right)$$
$$+ d(\lambda) \left(\frac{q}{p}\right)^{\sum_{j=1}^{N-1}(n_{j-1}-n_j)\theta_{jN}} \prod_{j=1}^{n_{N-1}} f\left(\lambda, \lambda^{(N-1)}_j\right). \tag{5.32}$$

The unwanted terms disappear when the set of unknown numbers $\left\{\lambda^{(n)}_l\right\}^{n_l}_{j=1}$ $(1 \leqslant l \leqslant N-1)$ satisfies the following Bethe equations, which are also derived by imposing the pole-free conditions on the eigenvalue formula:

$$\left(\frac{q}{p}\right)^{(L-n_1)\theta_{12}} d\left(\lambda_j^{(1)}\right) = -\left(\frac{q}{p}\right)^{-\overline{n}_1+\overline{n}_2} \prod_{k=1}^{n_1} \frac{f\left(\lambda_k^{(1)}, \lambda_j^{(1)}\right)}{f\left(\lambda_j^{(1)}, \lambda_k^{(1)}\right)} \prod_{k=1}^{n_2} \frac{1}{f\left(\lambda_k^{(2)}, \lambda_j^{(1)}\right)},$$

$$\left(\frac{q}{p}\right)^{\sum_{j=1}^{l}(n_{j-1}-n_j)\theta_{jl+1}-\sum_{j=1}^{l-1}(n_{j-1}-n_j)\theta_{jl}}$$

$$= -\left(\frac{q}{p}\right)^{-\overline{n}_l+\overline{n}_{l+1}} \prod_{k=1}^{n_l} \frac{f\left(\lambda_k^{(l)}, \lambda_j^{(l)}\right)}{f\left(\lambda_j^{(l)}, \lambda_k^{(l)}\right)} \frac{\prod_{k=1}^{n_{l-1}} f\left(\lambda_j^{(l)}, \lambda_k^{(l-1)}\right)}{\prod_{k=1}^{n_{l+1}} f\left(\lambda_k^{(l+1)}, \lambda_j^{(l)}\right)} \quad (2 \leqslant l \leqslant N-2),$$

$$\left(\frac{q}{p}\right)^{\sum_{j=1}^{N-1}(n_{j-1}-n_j)\theta_{jN}-\sum_{j=1}^{N-2}(n_{j-1}-n_j)\theta_{jN-1}}$$

$$= -\left(\frac{q}{p}\right)^{-\overline{n}_{N-1}} \prod_{k=1}^{n_{N-1}} \frac{f\left(\lambda_k^{(N-1)}, \lambda_j^{(N-1)}\right)}{f\left(\lambda_j^{(N-1)}, \lambda_k^{(N-1)}\right)} \prod_{k=1}^{n_{N-2}} f\left(\lambda_j^{(N-1)}, \lambda_k^{(N-2)}\right). \tag{5.33}$$

Inserting expression (5.32) into (5.8), one finds the spectrum of the Hamiltonian:

$$E = \frac{\partial}{\partial\lambda} \ln \Lambda(\lambda)\bigg|_{\lambda=0} = \sum_{j=1}^{n_1} \frac{\left(1 - p\lambda_j^{(1)}\right)\left(1 - q\lambda_j^{(1)}\right)}{\lambda_j^{(1)}}. \tag{5.34}$$

Though the explicit form of the Bethe equation (5.33) and its solutions depend on the nesting order in general, the spectrum of the Hamiltonian (5.34), of course, does not depend on it.

5.1.2. Completeness of the Bethe ansatz. In this sub-subsection we exclusively consider the standard nesting order $a_j = j (1 \leqslant j \leqslant N)$. Let us recall our setting and definitions. We consider the transfer matrix $T(\lambda)$ (5.5) acting on the sector $V(m)$ (see (2.12)). The data $m = (m_1, \dots, m_N) \in \mathbb{Z}_{\geqslant 0}^N$ specify the number m_j of the particles of the jth kind and $m_1 + \cdots + m_N = L$. The sector m is basic if it has the form $m = (m_1, \dots, m_n, 0, \dots, 0)$ with $m_1, \dots, m_n$ all being positive for some $1 \leqslant n \leqslant L$. The basic sectors are labeled either with $\mathcal{M}$ (2.15) or $\mathcal{S}$ (4.2) by the one to one correspondence $\mathcal{M} \ni m \leftrightarrow \mathfrak{s} \in \mathcal{S}$ (4.3). For a basic sector m, we write $V(m)$ also as $V_\mathfrak{s}$ as in (4.8). $Y_\mathfrak{s}$ is the genuine component of $V_\mathfrak{s}$ defined in (4.22). $\mathrm{Spec}^\circ(\mathfrak{s})$ is the multiset of genuine eigenvalues of H in $Y_\mathfrak{s}$ (4.24).

Conjecture 5.1. *Suppose that $p \neq q$ are generic. Then for any sector $V(m)$ which is not necessarily basic, there exist $d = \dim V(m)$ distinct polynomials $\Lambda_1(\lambda), \dots, \Lambda_d(\lambda)$ in λ such that* $\det(\zeta - T(\lambda)) = \prod_{g=1}^{d}(\zeta - \Lambda_g(\lambda))$.

We call $\Lambda_1(\lambda), \Lambda_2(\lambda), \dots, \Lambda_d(\lambda)$ the *eigen-polynomials* of $T(\lambda)$. (It should not be confused with the characteristic polynomial $\det(\zeta - T(\lambda))$.) A direct consequence of conjecture 5.1 is that the transfer matrix $T(\lambda)$ hence the Hamiltonian H are diagonalizable in arbitrary sectors. (At $p = q$, the diagonalizability still holds but $\Lambda_g(\lambda)$'s are no longer distinct due to degeneracy caused by $sl(N)$-invariance.)

Now we turn to the completeness of the Bethe ansatz. In the remainder of this subsection and section 5.2.1, by the Bethe equations we mean the *polynomial equations* on $\left\{\lambda_j^{(l)} \middle| 1 \leqslant l \leqslant N-1, 1 \leqslant j \leqslant n_j\right\}$ obtained from (5.33) by multiplying a polynomial in them so that the resulting two sides do not share a nontrivial common factor. We say that a set of complex numbers $\left\{\lambda_j^{(l)}\right\}$ is a Bethe root if it satisfies the Bethe equations. Bethe roots $\left\{\lambda_j^{(l)}\right\}$ and $\left\{\lambda_j^{(l)\prime}\right\}$ are identified if $\lambda_j^{(l)} = \lambda_{k_j}^{(l)\prime}$ for some permutation $k_1, \dots, k_{n_l}$ of $1, \dots, n_l$ for each l. We say that a Bethe root $\{\lambda_j^{(l)}\}$ is *regular* if none of them is equal to $1/p$ and the two sides of any Bethe equation are nonzero. Using the same notation as in conjecture 5.1, we propose the following conjecture.

Conjecture 5.2 (Completeness). *Suppose $p \neq q$ are generic.*

(1) For any sector, all the eigen-polynomials $\Lambda_g(\lambda)$ are expressed in the form (5.32) in terms of some Bethe root.
(2) For a basic sector $\mathfrak{s} \in \mathcal{S}$, there exist exactly $\dim Y_{\mathfrak{s}}$ *regular Bethe roots and the associated* $\dim Y_{\mathfrak{s}}$ *eigen-polynomials among $\Lambda_1(\lambda), \ldots, \Lambda_d(\lambda)$.*
(3) The $\dim Y_{\mathfrak{s}}$ *eigen-polynomials in (2) give* $\mathrm{Spec}^{\circ}(\mathfrak{s})$ *by the logarithmic derivative (5.34).*

In view of section 4.5, it is natural to call the (conjectural) $\dim Y_{\mathfrak{s}}$ eigen-polynomials in conjecture 5.2 (2) the *genuine* eigen-polynomials of the basic sector $\mathfrak{s}$. Then conjecture 5.2 (3) is rephrased as claiming that the spectrum $\mathrm{Spec}(\mathfrak{s})$ and the genuine spectrum $\mathrm{Spec}^{\circ}(\mathfrak{s})$ are obtained by the logarithmic derivatives of the $\dim V_{\mathfrak{s}}$ eigen-polynomials and the $\dim Y_{\mathfrak{s}}$ genuine eigen-polynomials, respectively.

Some examples supporting conjectures 5.1 and 5.2 are presented in appendix C. In conjecture 5.2 (1), the Bethe roots corresponding to a non-genuine eigen-polynomial are not necessarily unique. See the 2nd and the 3rd examples from the last in appendix C. We expect that the Bethe vectors associated with the regular Bethe roots form a basis of $Y_{\mathfrak{s}}$.

Theorem 4.5 and conjecture 5.2 (2) bear some analogy with the $sl(N)$-invariant Heisenberg chain ($p = q$). There, the number of the Bethe roots is conjecturally the Kostka numbers [KKR] and the spectral embedding is induced by $sl(N)$ actions. Here, the analogous roles are played by $\dim Y_{\mathfrak{s}}$ and $\overleftarrow{\varphi}_{\mathfrak{s},\mathfrak{t}}$, respectively.

5.2. *Properties of the spectrum*

Now we derive some consequences of the eigenvalue formula (5.32), (5.33) and (5.34). Sections 5.2.3 and 5.2.4 are reviews of known derivation for reader's convenience.

5.2.1. Spectral inclusion property. First we rederive the spectral inclusion property (theorem 4.5) in the Bethe ansatz framework. Consider the sector $\mathfrak{t} \in \mathcal{S}$ where the number of particles of the jth kind is $m_j \geqslant 1$ for any j. In the notation (4.3), $\mathfrak{t}$ reads

$$\begin{aligned} \mathfrak{t} &= \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_{N-1}\} \\ &\leftrightarrow 1^{m_1} 2^{m_2} \cdots N^{m_N} = a_1^{m_{a_1}} \cdots a_N^{m_{a_N}}, \end{aligned} \tag{5.35}$$

where $m_1 + \cdots + m_N = L \geqslant 2$. Set

$$a_N = a_{N-1} + 1, \qquad \lambda_j^{(N-1)} \to \frac{1}{p} \quad (1 \leqslant j \leqslant n_{N-1}). \tag{5.36}$$

Due to the relations

$$\overline{n}_{N-1} = 0, \qquad f\left(\frac{1}{p}, \lambda\right) = \frac{p}{q} f\left(\lambda, \frac{1}{p}\right) = 1 \qquad \text{for} \quad \lambda \neq \frac{1}{p}, \tag{5.37}$$

the following reduction relation holds:

$$\Lambda(\lambda) = \overline{\Lambda}(\lambda) + d(\lambda) \left(\frac{q}{p}\right)^{\sum_{j=1}^{N-2}(n_{j-1}-n_j)\theta_{jN} + n_{N-1}}. \tag{5.38}$$

Here $\overline{\Lambda}(\lambda)$ stands for the eigenvalue formula of the $(N-2)$-species case in the sector

$$\tilde{a}_1^{m_{a_1}} \cdots \tilde{a}_{N-1}^{m_{a_{N-1}}+m_{a_N}} = 1^{m_1} \cdots a_{N-1}^{m_{a_{N-1}}+m_{a_N}} a_N^{m_{a_N+1}} \cdots (N-1)^{m_N} \leftrightarrow \mathfrak{t}\backslash\{m_1 + \cdots + m_{a_{N-1}}\} = \mathfrak{u}, \tag{5.39}$$

where

$$\tilde{a}_j = \begin{cases} a_j & \text{for} \quad a_j < a_N \\ a_j - 1 & \text{for} \quad a_j \geqslant a_N \end{cases} \qquad (1 \leqslant j \leqslant N-1). \tag{5.40}$$

If $\{\lambda_j^{(l)} | 1 \leqslant j \leqslant n_l, 1 \leqslant l \leqslant N-1\}$ is a solution of the Bethe equation in the sector $\mathfrak{t}$, so is $\{\lambda_j^{(l)} | 1 \leqslant j \leqslant n_l, 1 \leqslant l \leqslant N-2\}$ left after the substitution (5.36) in the sector $\mathfrak{u}$. This is because the last Bethe equation in (5.33) for $(N-1)$-species case becomes trivial, or alternatively one may say that the resulting $(N-2)$-species Bethe equation guarantees that $\overline{\Lambda}(\lambda)$ is pole-free.

Inserting (5.38) into (5.34), and using $d(0) = d'(0) = 0$, one thus finds the set of eigenvalues of the Hamiltonian for the sector $\mathfrak{t}$ that includes that for the sector $\mathfrak{u}$. Applying this argument repeatedly, one can see

$$\mathrm{Spec}(\mathfrak{s}) \subset \mathrm{Spec}(\mathfrak{t}) \qquad \text{for} \quad \mathfrak{s} \subset \mathfrak{t}. \tag{5.41}$$

That is theorem 4.5. Since the solutions of the Bethe equation (5.33) depend on the nesting order, the set of solutions characterizing the above $\mathrm{Spec}(\mathfrak{s})$ is, in general, not included in the original set of solutions characterizing $\mathrm{Spec}(\mathfrak{t})$.

5.2.2. Stationary state. One of the direct consequences of the above property is that the stationary state $E = 0$ for an arbitrary sector $\mathfrak{t}$ (5.35) is given by setting all the Bethe roots to $1/p$, i.e.

$$\lambda_j^{(l)} \to \frac{1}{p} \quad (1 \leqslant j \leqslant n_l, 1 \leqslant l \leqslant N). \tag{5.42}$$

It immediately follows that the eigen-polynomial of the stationary state is given by

$$\Lambda(\lambda) = 1 + \sum_{k=1}^{N-1} \left(\frac{q}{p}\right)^{n_k} d(\lambda), \tag{5.43}$$

where n_k is defined by (5.18), and we consider the standard nesting order $a_j = j (1 \leqslant j \leqslant N)$.

On the other hand, in the framework of the Bethe ansatz, the calculation of the corresponding eigenstate is rather cumbersome. This is sketched in appendix B.

5.2.3. KPZ universality class. From section 5.2.1, one immediately sees that the set of spectra for the sector $\mathfrak{t}$ (5.35) includes those for sectors consisting of single particles:

$$\mathrm{Spec}(\mathfrak{s}_l) \subset \mathrm{Spec}(\mathfrak{t}), \qquad \mathfrak{s}_l = \{m_1 + \cdots + m_l\} \leftrightarrow 1^{m_1+\cdots+m_l} 2^{m_{l+1}+\cdots+m_N} \quad (1 \leqslant l \leqslant N-1). \tag{5.44}$$

As discussed in section 3, the relaxation spectra characterizing the universality class are the eigenvalues in the sector $\mathfrak{s}_l$, whose real parts have the second largest value. As described below or in section 3, these eigenvalues form a complex-conjugate pair. We denote them by $E_l^{\pm}$ hereafter. The Bethe equation (5.33) describing $\mathrm{Spec}(\mathfrak{s}_l)$ reduces to

$$(p\lambda_j)^L = (-1)^{n_1-1} \prod_{k=1}^{n_1} \frac{1 - (p+q)\lambda_j + pq\lambda_j\lambda_k}{1 - (p+q)\lambda_k + pq\lambda_j\lambda_k}, \qquad n_1 = L - (m_1 + \cdots + m_l), \tag{5.45}$$

where we set the nesting order as $(a_1, a_2) = (1, 2)$. Since the spectrum of the Hamiltonian is invariant under the change of the nesting order, and under the transformation $p \leftrightarrow q$,[8] it is enough to consider the case $p \geqslant q$ and $n_1 \leqslant L/2$.

For $p \neq q$, $E_l^{\pm}$ characterizes the KPZ universality class. The corresponding solutions to (5.45) are determined as follows [K]. Changing the variable λ as

$$p\lambda_j = \frac{1-x_j}{1-\mathrm{e}^{-2\eta}x_j}, \qquad \frac{p}{q} = \mathrm{e}^{-2\eta}, \tag{5.46}$$

we modify (5.45) and (5.34):

$$\begin{aligned} &\left(\frac{1-x_j}{1-\mathrm{e}^{-2\eta}x_j}\right)^L = (-1)^{n_1-1}\prod_{k=1}^{n_1}\frac{x_j-\mathrm{e}^{-2\eta}x_k}{x_k-\mathrm{e}^{-2\eta}x_j}, \\ &E = (p-q)\sum_{j=1}^{n_1}\left(\frac{x_j}{1-x_j} - \frac{\mathrm{e}^{-2\eta}x_j}{1-\mathrm{e}^{-2\eta}x_j}\right). \end{aligned} \tag{5.47}$$

The meaning of η in the above is revealed in section 5.3. Taking logarithm on both sides, one has

$$\frac{L}{2\pi\mathrm{i}}\log\frac{1-x_j}{x_j^{\rho}(1-\mathrm{e}^{-2\eta}x_j)} = I_j - \frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{n_1}\left(\log x_k - \log\frac{1-\mathrm{e}^{-2\eta}x_k/x_j}{1-\mathrm{e}^{-2\eta}x_j/x_k}\right), \tag{5.48}$$

where $\rho = n_1/L$ and $I_j \in \mathbb{Z} + \frac{1+(-1)^{n_1}}{4}$. In fact, for sufficiently large n_1 and L, the following choice

$$I_j^- = \begin{cases} -\dfrac{n_1+1}{2} + j & \text{for} \quad 1 \leqslant j \leqslant n_1 - 1 \\ \dfrac{n_1+1}{2} & \text{for} \quad j = n_1, \end{cases} \qquad I_j^+ = -I_j^- \tag{5.49}$$

gives the solution corresponding to $E_l^{\pm}$.

By carefully taking into account finite-size corrections, the asymptotic form of $E_l^{\pm}$ for $L \gg 1$ is determined as

$$E_l^{\pm} = \pm 2|(p-q)(1-2\rho)|\pi\mathrm{i}L^{-1} - 2C|p-q|\sqrt{\rho(1-\rho)}L^{-\frac{3}{2}} + O(L^{-2}), \tag{5.50}$$

where $C = 6.509\,189\,337\,94\ldots$ [K]. Thus we conclude that the system for $p \neq q$ belongs to the KPZ universality class whose dynamical exponent is $z = 3/2$.

5.2.4. EW universality class. For $p = q$, the set of eigenvalues of the Hamiltonian for an arbitrary sector $\mathfrak{t}$ contains the relaxation spectrum corresponding to the 'one-magnon' states. This can be seen by setting all the roots in (5.45) except for λ_1 to $1/p$. Thus $E_l^{\pm}$ are given by the second largest eigenvalues for this one-magnon states, and obviously do not depend on l. The Bethe ansatz equation determining the unknown λ_1 simply reduces to

$$(p\lambda_1)^L = 1. \tag{5.51}$$

Solving this and substituting the solutions

$$p\lambda_1 = \exp\left(\pm\frac{2\pi k\mathrm{i}}{L}\right) \qquad 1 \leqslant k \leqslant \frac{L}{2} \tag{5.52}$$

into (5.34), we have $E = -4p\sin^2(2\pi k/L)$. Obviously the case $k = 1$ gives the second largest eigenvalues:

$$E_l^{\pm} = -4p\sin^2\left(\frac{\pi}{L}\right) = -4p\pi^2L^{-2} + O(L^{-4}), \tag{5.53}$$

which gives the EW exponent $z = 2$.

[8] This can easily be seen from the fact that the Bethe equation (5.33) is invariant under the transformation $p \leftrightarrow q$ and $(a_1, \ldots, a_N) \leftrightarrow (N+1-a_1, \ldots, N+1-a_N)$.

5.3. Parameterization with difference property

Here we present the Bethe ansatz results in a more conventional parameterization [Sc, PS] with the spectral parameter having a difference property.

First we treat the one-species case ($N = 2$) whose spectrum is given by (5.34) via the Bethe ansatz (5.45), where $0 \leqslant n_1 \leqslant L$ and the nesting order is $(a_1, a_2) = (1, 2)$. Changing the variables as

$$p\lambda_j^{(1)} \to \exp(\mathrm{i}p_j + \eta), \qquad \frac{q}{p} = \mathrm{e}^{-2\eta}, \tag{5.54}$$

we transform (5.34) and (5.45) to

$$\begin{aligned} E &= 2\sqrt{pq}\sum_{j=1}^{n_1}(\cos p_j - \Delta), \\ \mathrm{e}^{\mathrm{i}Lp_j} &= (-1)^{n_1-1}\,\mathrm{e}^{-\eta L}\prod_{k=1}^{n_1}\frac{1+\mathrm{e}^{\mathrm{i}(p_j+p_k)}-2\Delta\mathrm{e}^{\mathrm{i}p_j}}{1+\mathrm{e}^{\mathrm{i}(p_j+p_k)}-2\Delta\mathrm{e}^{\mathrm{i}p_k}}, \qquad \Delta = \cosh\eta. \end{aligned} \tag{5.55}$$

This is nothing but the eigenvalue of the Hamiltonian for the XXZ chain threaded by a 'magnetic flux' $-\mathrm{i}\eta L$:

$$H = \sqrt{pq}\sum_{k\in\mathbb{Z}_L}\left\{\mathrm{e}^{\eta}\sigma_k^+\sigma_{k+1}^- + \mathrm{e}^{-\eta}\sigma_{k+1}^+\sigma_k^- + \frac{\Delta}{2}\left(\sigma_k^z\sigma_{k+1}^z - 1\right)\right\}. \tag{5.56}$$

The variable p_j in (5.55) is called the quasi-momentum of the Bethe wavefunction. Introducing the transformation (see [T] for example)

$$\begin{aligned} \widetilde{p}(u) &:= \frac{1}{\mathrm{i}}\log\frac{\mathrm{sh}\frac{\eta}{2}u}{\mathrm{sh}\frac{\eta}{2}(u+2)}, \qquad & p_j &= \widetilde{p}\big(\mathrm{i}u_j^{(1)} - 1\big), \\ \widetilde{\lambda}(u) &:= \frac{1}{p}\exp(\mathrm{i}\widetilde{p}(u) + \eta), \qquad & \lambda_j^{(1)} &= \widetilde{\lambda}\big(\mathrm{i}u_j^{(1)} - 1\big), \end{aligned} \tag{5.57}$$

we rewrite (5.55) in terms of the 'rapidities' $u_j^{(1)}$:

$$E = 2\sqrt{pq}\sum_{j=1}^{n_1}\frac{\mathrm{sh}^2\eta}{\mathrm{ch}\big(\eta u_j^{(1)}\big) - \mathrm{ch}\,\eta}, \qquad \phi\big(u_j^{(1)}\big) = -\mathrm{e}^{-\eta L}\frac{q_1\big(u_j^{(1)} + 2\mathrm{i}\big)}{q_1\big(u_j^{(1)} - 2\mathrm{i}\big)}, \tag{5.58}$$

where the two functions $\phi(u)$ and $q_1(u)$ are defined by

$$\phi(u) = \left(\frac{\sin\frac{\eta}{2}(u+\mathrm{i})}{\sin\frac{\eta}{2}(u-\mathrm{i})}\right)^L, \qquad q_i(u) = \prod_{j=1}^{n_i}\sin\frac{\eta}{2}\big(u - u_j^{(i)}\big). \tag{5.59}$$

Applying the momentum-rapidity transformation (5.57) to the R-matrix (5.2) (we write $R(\lambda(u)) = \widetilde{R}(u)$), we find that the non-zero elements of $\widetilde{R}(u)$ can be written as

$$\widetilde{R}_{\alpha\alpha}^{\alpha\alpha}(u) = 1, \quad \widetilde{R}_{\alpha\beta}^{\alpha\beta}(u) = \begin{cases} \dfrac{\mathrm{e}^{-\eta}\mathrm{sh}\frac{\eta}{2}u}{\mathrm{sh}\frac{\eta}{2}(u+2)} & \text{for } \alpha < \beta, \\ \dfrac{\mathrm{e}^{\eta}\mathrm{sh}\frac{\eta}{2}u}{\mathrm{sh}\frac{\eta}{2}(u+2)} & \text{for } \alpha > \beta, \end{cases} \quad \widetilde{R}_{\alpha\beta}^{\beta\alpha}(u) = \begin{cases} \dfrac{\mathrm{e}^{\frac{\eta}{2}u}\mathrm{sh}\,\eta}{\mathrm{sh}\frac{\eta}{2}(u+2)} & \text{for } \alpha < \beta, \\ \dfrac{\mathrm{e}^{-\frac{\eta}{2}u}\mathrm{sh}\,\eta}{\mathrm{sh}\frac{\eta}{2}(u+2)} & \text{for } \alpha > \beta, \end{cases} \tag{5.60}$$

where $\alpha, \beta \in \{1, 2\}$ in the present case. Up to the asymmetric factors $\mathrm{e}^{\pm\eta}$ and $\mathrm{e}^{\pm\eta u/2}$, these are the Boltzmann weights for the well-known six vertex model [Ba] associated with the quantum group $U_q(\widehat{sl}(2))$. For the gauge factors, see [PS, OY]. The *R*-matrix (5.60) satisfies the Yang–Baxter equation

$$\widetilde{R}_{23}(u_2)\widetilde{R}_{13}(u_1)\widetilde{R}_{12}(u_1-u_2)=\widetilde{R}_{12}(u_1-u_2)\widetilde{R}_{13}(u_1)\widetilde{R}_{23}(u_2), \tag{5.61}$$

which possesses the difference property. The Hamiltonian (5.56) is expressed as the logarithmic derivative of the transfer matrix $\widetilde{T}(u)$ (cf (5.7)):

$$\begin{aligned}&\widetilde{T}(u)=\mathrm{tr}_{W_0}[\widetilde{R}_{0L}(\mathrm{i}u-1)\cdots\widetilde{R}_{01}(\mathrm{i}u-1)],\\ &H=-\frac{2\mathrm{i}\sqrt{pq}\,\mathrm{sh}\,\eta}{\eta}\frac{\partial}{\partial u}\ln\widetilde{T}(u)\Big|_{u=-\mathrm{i}},\end{aligned} \tag{5.62}$$

where $N=2$ in the present case. Noting that

$$\begin{aligned}&d(\lambda(\mathrm{i}u-1))=(p\lambda(\mathrm{i}u-1))^L=\mathrm{e}^{\eta L}\phi(u),\\ &f(\lambda(\mathrm{i}u-1),\lambda(\mathrm{i}v-1))=\frac{1}{p\xi(\lambda(\mathrm{i}u-1),\lambda(\mathrm{i}v-1))}=\mathrm{e}^{-\eta}\frac{\sin\frac{\eta}{2}(u-v-2\mathrm{i})}{\sin\frac{\eta}{2}(u-v)},\end{aligned} \tag{5.63}$$

the eigenvalue of the $\widetilde{T}(u)$ for the nesting order $(a_1,a_2)=(1,2)$ is given by

$$\widetilde{\Lambda}(u)=\frac{q_1(u+2\mathrm{i})}{q_1(u)}\,\mathrm{e}^{-\eta n_1}+\phi(u)\frac{q_1(u-2\mathrm{i})}{q_1(u)}\mathrm{e}^{\eta L-\eta n_1}, \tag{5.64}$$

via the Bethe equation (5.58).

The extension to the general $(N-1)$-species case is straightforward. We just let the local states α,β in (5.60) range over $\alpha,\beta\in\{1,\ldots,N\}$. Finally, we write the explicit form of the eigenvalues for an arbitrary nesting order:

$$\begin{aligned}\widetilde{\Lambda}(u)=&\frac{q_1(u+2\mathrm{i})}{q_1(u)}\,\mathrm{e}^{-\eta(n_1-2\overline{n}_1)}\\ &+\phi(u)\sum_{k=1}^{N-2}\frac{q_k(u-2\mathrm{i})}{q_k(u)}\frac{q_{k+1}(u+2\mathrm{i})}{q_{k+1}(u)}\,\mathrm{e}^{\eta L-\eta(n_k+n_{k+1}+2\sum_{j=1}^{k}(n_{j-1}-n_j)\theta_{jk+1}-2\overline{n}_{k+1})}\\ &+\phi(u)\frac{q_{N-1}(u-2\mathrm{i})}{q_{N-1}(u)}\,\mathrm{e}^{\eta L-\eta(n_{N-1}+2\sum_{j=1}^{N-1}(n_{j-1}-n_j)\theta_{jN}}.\end{aligned} \tag{5.65}$$

Correspondingly the Bethe equation (5.33) is transformed to

$$\begin{aligned}&\mathrm{e}^{-2\eta(L-n_1)\theta_{12}}\phi\big(u_j^{(1)}\big)=-\mathrm{e}^{-\eta L+\eta(n_2+2(\overline{n}_1-\overline{n}_2))}\frac{q_1\big(u_j^{(1)}+2\mathrm{i}\big)}{q_1\big(u_j^{(1)}-2\mathrm{i}\big)}\frac{q_2\big(u_j^{(1)}\big)}{q_2\big(u_j^{(1)}+2\mathrm{i}\big)},\\ &\mathrm{e}^{-2\eta(\sum_{j=1}^{l}(n_{j-1}-n_j)\theta_{jl+1}-\sum_{j=1}^{l-1}(n_{j-1}-n_j)\theta_{jl})}\\ &\quad=-\mathrm{e}^{\eta(n_{l+1}-n_{l-1}+2(\overline{n}_l-\overline{n}_{l+1}))}\frac{q_l\big(u_j^{(l)}+2\mathrm{i}\big)}{q_l\big(u_j^{(l)}-2\mathrm{i}\big)}\frac{q_{l-1}\big(u_j^{(l)}-2\mathrm{i}\big)}{q_{l-1}\big(u_j^{(l)}\big)}\frac{q_{l+1}\big(u_j^{(l)}\big)}{q_{l+1}\big(u_j^{(l)}+2\mathrm{i}\big)}\\ &\quad(2\leqslant l\leqslant N-2),\\ &\mathrm{e}^{-2\eta(\sum_{j=1}^{N-1}(n_{j-1}-n_j)\theta_{jN}-\sum_{j=1}^{N-2}(n_{j-1}-n_j)\theta_{jN-1})}\\ &\quad=-\mathrm{e}^{\eta(-n_{N-2}+2\overline{n}_{N-1})}\frac{q_{N-1}\big(u_j^{(N-1)}+2\mathrm{i}\big)}{q_{N-1}\big(u_j^{(N-1)}-2\mathrm{i}\big)}\frac{q_{N-2}\big(u_j^{(N-1)}-2\mathrm{i}\big)}{q_{N-2}\big(u_j^{(N-1)}\big)}.\end{aligned} \tag{5.66}$$

The spectrum of the Hamiltonian H is then determined by

$$E = 2\sqrt{pq}\sum_{j=1}^{n_1}\frac{\mathrm{sh}^2\eta}{\mathrm{ch}\big(\eta u_j^{(1)}\big)-\mathrm{ch}\,\eta}. \tag{5.67}$$

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Appendix A. Proof of theorem 4.9

A.1. Möbius inversion

The power set $\mathcal{S}$ (4.2) is equipped with the natural poset structure whose partial order is just $\subseteq$. In this appendix the partial order in $\mathcal{M}$ (2.15) induced via (4.3) will be denoted by $\preceq$. Thus one has $(4) \preceq (1,3) \preceq (1,2,1) \preceq (1,1,1,1)$, etc for $L = 4$. The description of $\preceq$ in $\mathcal{M}$ is pretty simple. In fact, those m' satisfying $m' \preceq m = (m_1,\ldots,m_n)$ are obtained from m by successive contractions

$$(\ldots,m_j,m_{j+1},\ldots) \mapsto (\ldots,m_j+m_{j+1},\ldots). \tag{A.1}$$

Let $\zeta = \big(\zeta(\mathfrak{s}',\mathfrak{s})\big)_{\mathfrak{s}',\mathfrak{s}\in\mathcal{S}}$ be the $|\mathcal{S}|\times|\mathcal{S}|$ matrix defined by

$$\zeta(\mathfrak{s}',\mathfrak{s}) = \begin{cases}1 & \mathfrak{s}'\subseteq\mathfrak{s},\\ 0 & \text{otherwise.}\end{cases} \tag{A.2}$$

Since ζ is a triangular matrix whose diagonal elements are all 1, it has the inverse $\mu = (\mu(\mathfrak{s}',\mathfrak{s}))_{\mathfrak{s}',\mathfrak{s}\in\mathcal{S}}$. μ is called the Möbius function of $\mathcal{S}$, and is again a triangular (i.e., $\mu(\mathfrak{s}',\mathfrak{s}) = 0$ unless $\mathfrak{s}'\subseteq\mathfrak{s}$) integer matrix.

Suppose $f, g : \mathcal{S}\to\mathbb{C}$ are the functions on $\mathcal{S}$. By the definition, the two relations

$$f(\mathfrak{s}) = \sum_{\mathfrak{s}'\subseteq\mathfrak{s}} g(\mathfrak{s}'), \qquad g(\mathfrak{s}) = \sum_{\mathfrak{s}'\subseteq s}\mu(\mathfrak{s}',\mathfrak{s})f(\mathfrak{s}') \quad (\mathfrak{s}\in\mathcal{S}) \tag{A.3}$$

are equivalent, where the latter is the Möbius inversion formula. In a matrix notation, they are just $f = g\zeta$ and $g = f\mu$. In particular the sum involving $\mu(\mathfrak{s}',\mathfrak{s})$ can be restricted to $\mathfrak{s}'\subseteq\mathfrak{s}$. The Möbius function contains all the information on the poset structure. In our case of the power set $\mathcal{S}$, it is a classical result (the inclusion–exclusion principle) that

$$\mu(\mathfrak{s}',\mathfrak{s}) = (-1)^{\sharp\mathfrak{s}'-\sharp\mathfrak{s}}, \tag{A.4}$$

where $\sharp\mathfrak{s}$ denotes the cardinality of $\mathfrak{s}$.

The Möbius inversion formulae (A.3) and (A.4) on $\mathcal{S}$ can be translated into those on $\mathcal{M}$ via the bijective correspondence (4.3). The result reads as follows:

$$f(m) = \sum_{m'\preceq m} g(m') \quad (m\in\mathcal{M}), \tag{A.5}$$

$$g(m_1,\ldots,m_n) = \sum(-1)^{n-l}f(i_1,\ldots,i_l) \quad ((m_1,\ldots,m_n)\in\mathcal{M}), \tag{A.6}$$

where the sum in (A.6) extends over $(i_1, \ldots, i_l) \in \mathcal{M}$ such that $(i_1, \ldots, i_l) \preceq (m_1, \ldots, m_n)$. (We have written $g(m)$ with $m = (m_1, \ldots, m_n)$ as $g(m_1, \ldots, m_n)$ rather than $g((m_1, \ldots, m_n))$, and similarly for f.)

For $m = (m_1, \ldots, m_n) \in \mathcal{M}$ corresponding to $\mathfrak{s} \in \mathcal{S}$, we let $\overline{m}$ denote the element in $\mathcal{M}$ that corresponds to the complement $\bar{\mathfrak{s}} = \Omega \backslash \mathfrak{s} \in \mathcal{S}$. Thus for $L = 4$, $^{-}$ acts as the involution

$$\emptyset \rightleftarrows \{1,2,3\}, \quad \{1\} \rightleftarrows \{2,3\}, \quad \{2\} \rightleftarrows \{1,3\}, \quad \{3\} \rightleftarrows \{1,2\}$$

on $\mathcal{S}$, and similarly

$$(4) \rightleftarrows (1,1,1,1), \quad (1,3) \rightleftarrows (2,1,1), \quad (2,2) \rightleftarrows (1,2,1), \quad (3,1) \rightleftarrows (1,1,2)$$

on $\mathcal{M}$. It is an easy exercise to check

$$\overline{(m_1, \ldots, m_n)} = (1^{m_1-1}21^{m_2-2}21^{m_3-2}\ldots 21^{m_{n-1}-2}21^{m_n-1}) \in \mathcal{M}, \tag{A.7}$$

where '$a1^{-1}b$' should be understood as $a + b - 1$.

A.2. Theorem 4.9

We keep assuming the one to one correspondence (4.3) of the labels $m \in \mathcal{M}$ and $\mathfrak{s} \in \mathcal{S}$ and use the former. In view of (4.7) we have $\dim V_{\mathfrak{s}}^* = f(m)$ by the choice:

$$f(m) = \binom{L}{m_1, \ldots, m_n} := \frac{L!}{m_1! \cdots m_n!} \qquad \text{for} \quad m = (m_1, \ldots, m_n) \in \mathcal{M}. \tag{A.8}$$

Denote the $g(m)$ determined from this and (A.6) by

$$g(m) = \left\langle \begin{matrix} L \\ m_1, \ldots, m_n \end{matrix} \right\rangle. \tag{A.9}$$

Namely, we define

$$\left\langle \begin{matrix} L \\ m_1, \ldots, m_n \end{matrix} \right\rangle = \sum (-1)^{n-l} \binom{L}{i_1, \ldots, i_l} \qquad \text{for} \quad (m_1, \ldots, m_n) \in \mathcal{M}, \tag{A.10}$$

where the sum runs over $(i_1, \ldots, i_l) \in \mathcal{M}$ such that $(i_1, \ldots, i_l) \preceq (m_1, \ldots, m_n)$. From (4.7), (4.27), (A.5) and (A.8), we find $\dim X_{\mathfrak{s}}^* = g(m)$. The function $g(m)$ has the invariance

$$\left\langle \begin{matrix} L \\ m_1, \ldots, m_n \end{matrix} \right\rangle = \left\langle \begin{matrix} L \\ m_n, \ldots, m_1 \end{matrix} \right\rangle,$$

but it is *not* symmetric under general permutations of $m_1, \ldots, m_n$ in contrast to $f(m)$.

Now theorem 4.9 is translated into the following.

Theorem A.1.

$$g(m) = g(\overline{m}) \textit{ for any } m \in \mathcal{M}.$$

Example A.2. Take $m = (L)$ hence $\overline{m} = (1^L)$. Then $g(m) = \left\langle \begin{smallmatrix} L \\ L \end{smallmatrix} \right\rangle = \binom{L}{L} = 1$. On the other hand, $g(\overline{m})$ with $L = 3$ and 4 are calculated as

$$\binom{3}{1,1,1} - \binom{3}{2,1} - \binom{3}{1,2} + \binom{3}{3} = 1,$$

$$\binom{4}{1,1,1,1} - \binom{4}{2,1,1} - \binom{4}{1,2,1} - \binom{4}{1,1,2} + \binom{4}{3,1} + \binom{4}{2,2} + \binom{4}{1,3} - \binom{4}{4} = 1.$$

Example A.3. Take $L = 5$ and $m = (1, 2, 1, 1)$ hence $\overline{m} = (2, 3)$. Then one has

$$\left\langle {5 \atop 1,2,1,1} \right\rangle = \binom{5}{1,2,1,1} - \binom{5}{1,3,1} - \binom{5}{1,2,2} + \binom{5}{1,4} - \binom{5}{3,1,1} + \binom{5}{3,2} + \binom{5}{4,1} - \binom{5}{5}, \tag{A.11}$$

$$\left\langle {5 \atop 2,3} \right\rangle = \binom{5}{2,3} - \binom{5}{5}. \tag{A.12}$$

Both these sums yield 9.

A.3. Proof

We first generalize example A.2 to the following.

Lemma A.4.

$$\left\langle {L \atop L} \right\rangle = \left\langle {L \atop 1^L} \right\rangle \quad \textit{for any } L \geqslant 1.$$

Proof. The left-hand side is 1. From (A.10), the right-hand side is given by

$$\left\langle {L \atop 1^L} \right\rangle = \sum_{l=1}^{L} A_{L,l}, \qquad A_{L,l} = (-1)^{L-l} \sum_{i_1,\ldots,i_l} \binom{L}{i_1,\ldots,i_l},$$

where the latter sum extends over $i_1, \ldots, i_l \in \mathbb{Z}_{\geqslant 1}$ such that $i_1 + \cdots + i_l = L$. Thus we have the following evaluation of the generating functions:

$$\sum_{L\geqslant l} \frac{A_{L,l}}{L!} z^L = (-1)^l \sum_{i_1,\ldots,i_l} \frac{(-z)^{i_1}\cdots(-z)^{i_l}}{i_1!\cdots i_l!} = (1-\mathrm{e}^{-z})^l,$$

$$\sum_{L\geqslant 1} \frac{z^L}{L!} \sum_{l=1}^{L} A_{L,l} = \sum_{l\geqslant 1}\sum_{L\geqslant l} \frac{A_{L,l}}{L!} z^L = \sum_{l\geqslant 1} (1-\mathrm{e}^{-z})^l = \mathrm{e}^z - 1.$$

The last relation tells that $\sum_{l=1}^{L} A_{L,l} = 1$ for $L \geqslant 1$. □

In (A.11), note that the first line of the right-hand side equals $\binom{5}{1,4}\left\langle {4 \atop 2,1,1} \right\rangle$, whereas the second line is nothing but $-\left\langle {5 \atop 3,1,1} \right\rangle$; therefore one has

$$\left\langle {5 \atop 1,2,1,1} \right\rangle = \binom{5}{1,4} \left\langle {4 \atop 2,1,1} \right\rangle - \left\langle {5 \atop 3,1,1} \right\rangle.$$

The following lemma shows that such a decomposition holds generally.

Lemma A.5.

$$\left\langle {L \atop m_1, m_2, \ldots, m_n} \right\rangle = \binom{L}{m_1} \left\langle {L - m_1 \atop m_2, \ldots, m_n} \right\rangle - \left\langle {L \atop m_1 + m_2, m_3, \ldots, m_n} \right\rangle, \tag{A.13}$$

where $\binom{L}{m_1} = \binom{L}{m_1, L-m_1}$ *denotes the binomial coefficient.*

Proof. In forming $(i_1, \ldots, i_l)$ from $(m_1, \ldots, m_n)$ by the successive contractions (A.1), we classify the summands in (A.10) according to whether m_1 has been contracted to m_2 or not. If it is not contracted, then $i_1 = m_1$ always holds and the corresponding summands yield

$\binom{L}{m_1}\left\langle {L-m_1 \atop m_2,\ldots,m_n} \right\rangle$. The other summands correspond to the contracted case $i_1 \geqslant m_1+m_2$, whose contribution is $-\left\langle {L \atop m_1+m_2, m_3,\ldots,m_n} \right\rangle$. □

Lemma A.6.

$$\left\langle {L \atop 1^t, \pi} \right\rangle = \sum_{s=0}^{a} (-1)^{s+a} \binom{L}{s} \left\langle {L-s \atop a+1-s, 1^{t-a-1}, \pi} \right\rangle \quad (1 \leqslant a \leqslant t-1), \tag{A.14}$$

where π is an arbitrary array of positive integers summing up to $L-t$.

Proof. We employ the induction on a. The case $a=1$, i.e.,

$$\left\langle {L \atop 1^t, \pi} \right\rangle = -\left\langle {L \atop 2, 1^{t-2}, \pi} \right\rangle + \binom{L}{1} \left\langle {L-1 \atop 1^{t-1}, \pi} \right\rangle$$

follows from lemma A.5. Suppose that (A.14) holds for $a=a$. The substitution of (A.13) gives

$$\left\langle {L \atop 1^t, \pi} \right\rangle = \sum_{s=0}^{a} (-1)^{s+a} \binom{L}{s} \left[\binom{L-s}{a+1-s} \left\langle {L-a-1 \atop 1^{t-a-1}, \pi} \right\rangle - \left\langle {L-s \atop a+2-s, 1^{t-a-2}, \pi} \right\rangle \right]. \tag{A.15}$$

In the first term, the s-sum is taken as

$$\sum_{s=0}^{a} (-1)^{s+a} \binom{L}{s} \binom{L-s}{a+1-s} = \binom{L}{a+1} \sum_{s=0}^{a} (-1)^{s+a} \binom{a+1}{s} = \binom{L}{a+1}. \tag{A.16}$$

Thus (A.15) implies the $a \to a+1$ case of (A.14). □

Lemma A.7.

$$\left\langle {L \atop 1^{a+1}, \pi} \right\rangle + \left\langle {L \atop 1^{a-1}, 2, \pi} \right\rangle = \binom{L}{a} \left\langle {L-a \atop 1, \pi} \right\rangle,$$

where π is an arbitrary array of positive integers summing up to $L-a-1$.

Proof. The case $t=a+1$ in (A.14) reads

$$\left\langle {L \atop 1^{a+1}, \pi} \right\rangle = \sum_{s=0}^{a} (-1)^{s+a} \binom{L}{s} \left\langle {L-s \atop a+1-s, \pi} \right\rangle.$$

Similarly, by setting $t \to a-1$, $a \to a-2$ and $\pi \to (2,\pi)$ in (A.14), we obtain

$$\begin{aligned}\left\langle {L \atop 1^{a-1}, 2, \pi} \right\rangle &= \sum_{s=0}^{a-2} (-1)^{s+a} \binom{L}{s} \left\langle {L-s \atop a-1-s, 2, \pi} \right\rangle \\ &= \sum_{s=0}^{a-2} (-1)^{s+a} \binom{L}{s} \left[\binom{L-s}{a-1-s} \left\langle {L-a+1 \atop 2, \pi} \right\rangle - \left\langle {L-s \atop a+1-s, \pi} \right\rangle \right],\end{aligned}$$

where (A.13) has been applied in the second equality. The sum of the two expressions gives

$$\begin{aligned}\left\langle {L \atop 1^{a+1}, \pi} \right\rangle + \left\langle {L \atop 1^{a-1}, 2, \pi} \right\rangle &= -\binom{L}{a-1} \left\langle {L-a+1 \atop 2, \pi} \right\rangle + \binom{L}{a} \left\langle {L-a \atop 1, \pi} \right\rangle \\ &\quad + \left[\sum_{s=0}^{a-2} (-1)^{s+a} \binom{L}{s} \binom{L-s}{a-1-s} \right] \left\langle {L-a+1 \atop 2, \pi} \right\rangle.\end{aligned}$$

The s-sum in the last term is $\binom{L}{a-1}$ due to (A.16), completing the proof. □

Proof of theorem A.1. For $(m_1, \ldots, m_n) \in \mathcal{M}$, we are to show

$$\left\langle \begin{matrix} L \\ m_1, \ldots, m_n \end{matrix} \right\rangle = \left\langle \begin{matrix} L \\ 1^{m_1-1}21^{m_2-2} \ldots 1^{m_{n-1}-2}21^{m_n-1} \end{matrix} \right\rangle,$$

where the explicit form of the dual is taken from (A.7). We invoke the double induction on (L, n). The case $L = 1$ is trivially true. In addition, the case $(m_1, \ldots, m_n) = (L)$ has already been verified for all L in lemma A.4. We assume that the assertion for (L', n') is true for $L' < L$ with any n' and also for (L, n') with $n' < n$. Consider the decomposition (A.13). By the induction assumption, the two quantities $\langle {}^{\cdot}_{\cdot} \rangle$ on the right-hand side can be replaced with their dual. Then the relation to be proved becomes

$$\left\langle \begin{matrix} L \\ 1^{m_1-1}21^{m_2-2} \ldots 1^{m_{n-1}-2}21^{m_n-1} \end{matrix} \right\rangle$$
$$= \binom{L}{m_1} \left\langle \begin{matrix} L - m_1 \\ 1^{m_2-1}21^{m_3-2} \ldots 1^{m_{n-1}-2}21^{m_n-1} \end{matrix} \right\rangle - \left\langle \begin{matrix} L \\ 1^{m_1+m_2-1}21^{m_3-2} \ldots 1^{m_{n-1}-2}21^{m_n-1} \end{matrix} \right\rangle.$$

In terms of $\pi = (1^{m_2-2}21^{m_3-2} \ldots 1^{m_{n-1}-2}21^{m_n-1})$, this is expressed as

$$\left\langle \begin{matrix} L \\ 1^{m_1-1}, 2, \pi \end{matrix} \right\rangle = \binom{L}{m_1} \left\langle \begin{matrix} L - m_1 \\ 1, \pi \end{matrix} \right\rangle - \left\langle \begin{matrix} L \\ 1^{m_1+1}, \pi \end{matrix} \right\rangle.$$

The proof is completed by lemma A.7. □

Appendix B. Derivation of the stationary state

Here we sketch a procedure to derive the stationary state in the framework of the algebraic Bethe ansatz. Throughout this appendix, we set the nesting order as a standard one: $a_j = j (1 \leqslant j \leqslant N)$.

As seen in section 5.2.2, the eigenvalue of the transfer matrix for the stationary state can simply be calculated by setting all the Bethe roots equal to $1/p$. In contrast to the eigenvalue problem, the evaluation of the eigenstate is not trivial. This is caused by the B-operators such as (5.12) and (5.27) that approach zero as $\lambda \to 1/p$. Thus to obtain the state, we must normalize B-operators as

$$\widetilde{B}^{(j)}_{\alpha_k}(\lambda) := \frac{B^{(j)}_{\alpha_k}(\lambda)}{1 - p\lambda}, \qquad \alpha_k \in \{j+2, \ldots, N\} \quad (1 \leqslant k \leqslant n_{j+1}; 0 \leqslant j \leqslant N-2). \tag{B.1}$$

Note that $B^{(0)}(\lambda) := B(\lambda)$. First we consider the eigenstate $|\{\lambda^{(N-1)}\}\rangle$ of $T^{(N-2)}(\lambda|\{\lambda^{(N-2)}\})$. As shown in section 5.1, this state is constructed by a multiple action of $\widetilde{B}^{(N-2)}_{\alpha}$ on $|\mathrm{vac}^{(N-2)}\rangle = |N-1\rangle_1 \otimes \cdots \otimes |N-1\rangle_{n_{N-2}}$:

$$\begin{aligned}
|\{\lambda^{(N-1)}\}\rangle &= \lim_{\lambda^{(N-1)}_j \to 1/p} \widetilde{B}^{(N-2)}_N\big(\lambda^{(N-1)}_1\big) \cdots \widetilde{B}^{(N-2)}_N\big(\lambda^{(N-1)}_{n_{N-1}}\big)|\mathrm{vac}^{(N-2)}\rangle \\
&= \mathrm{const.} \sum_{1 \leqslant \gamma_1 < \cdots < \gamma_{n_{N-1}} \leqslant n_{N-2}} \left\{ \prod_{j=1}^{n_{N-1}} \frac{1 - q\lambda^{(N-2)}_{\gamma_j}}{1 - p\lambda^{(N-2)}_{\gamma_j}} \right\} \left(\bigotimes_{k=1}^{n_{N-2}} \Big|N - 1 + \sum_{j=1}^{n_{N-1}} \delta_{k\gamma_j}\Big\rangle_k \right) \\
&=: F^{(N-3)\alpha_1 \cdots \alpha_{n_{N-2}}} |\alpha_1\rangle_1 \otimes \cdots \otimes |\alpha_{n_{N-2}}\rangle_{n_{N-2}}, \alpha_j \in \{N-1, N\} (1 \leqslant j \leqslant n_{N-2}),
\end{aligned} \tag{B.2}$$

where $F^{(N-3)\alpha_1 \cdots \alpha_{n_{N-2}}}$ is the element of $|\{\lambda^{(N-1)}\}\rangle$, from which the eigenstate $|\{\lambda^{(N-2)}\}\rangle$ of $T^{(N-3)}(\lambda|\{\lambda^{(N-3)}\})$ can be constructed as

$$|\{\lambda^{(N-2)}\}\rangle = \lim_{\lambda^{(N-2)}_j \to 1/p} F^{(N-3)\alpha_1 \cdots \alpha_{n_{N-2}}} \widetilde{B}^{(N-3)}_{\alpha_1}\big(\lambda^{(N-2)}_1\big) \cdots \widetilde{B}^{(N-3)}_{\alpha_{n_{N-2}}}\big(\lambda^{(N-2)}_{n_{N-2}}\big)|\mathrm{vac}^{(N-3)}\rangle. \tag{B.3}$$

Note that $F^{(0)\alpha_1\cdots\alpha_{n_1}}$ denotes $F^{\alpha_1\cdots\alpha_{n_1}}$ defined by (5.17). Since the coefficient $F^{(N-3)\alpha_1\cdots\alpha_{n_{N-2}}}$ contains the term $\prod\big(1-p\lambda_j^{(N-2)}\big)^{-1}$, we cannot take the limit $\lambda_j^{(N-2)}\to 1/p$ $(1\leqslant j\leqslant n_{N-2})$ independently of j. To take this limit correctly, we solve for the roots $\lambda_j^{(N-2)}$ $(2\leqslant j\leqslant n_{N-2})$ in terms of $\lambda_1^{(N-2)}$ by using the Bethe equations (5.33):

$$\begin{aligned} d\big(\lambda_j^{(1)}\big) &= -\prod_{k=1}^{n_1}\frac{f\big(\lambda_k^{(1)},\lambda_j^{(1)}\big)}{f\big(\lambda_j^{(1)},\lambda_k^{(1)}\big)} && \text{for}\quad N=3,\\ \frac{1}{\prod_{k=1}^{n_{N-3}} f\big(\lambda_j^{(N-2)},\lambda_k^{(N-3)}\big)} &= -\prod_{k=1}^{n_{N-2}}\frac{f\big(\lambda_k^{(N-2)},\lambda_j^{(N-2)}\big)}{f\big(\lambda_j^{(N-2)},\lambda_k^{(N-2)}\big)} && \text{for}\quad N>3. \end{aligned} \tag{B.4}$$

In the above, we have put $\lambda_j^{(N-1)}=1/p$. To extract the behavior of $\lambda_j^{(N-2)}$ around the point $1/p$, we expand them in terms of $\lambda_1^{(N-2)}-1/p$ as

$$\lambda_j^{(N-2)} = \frac{1}{p}+\sum_{k=1}^{n_{N-2}} g_j^{(k)}\big(\lambda_1^{(N-2)}-1/p\big)^k + O\Big(\big(\lambda_1^{(N-2)}-1/p\big)^{n_{N-2}+1}\Big)\quad \text{for}\quad 1\leqslant j\leqslant n_{N-2}, \tag{B.5}$$

where $g_1^{(k)}=\delta_{1k}$. Inserting them into (B.4) and comparing the coefficients of each order, one obtains the set of equations determining the coefficients $g_j^{(k)}$. In the following, as an example, we write the equations determining the first three coefficients:

$$\begin{aligned} b^{(1)} &= (-1)^{n_{N-2}+1}\prod_{k=1}^{n_{N-2}}\frac{c_{jk}^{(1)}}{c_{kj}^{(1)}},\qquad b^{(2)} = b^{(1)}\sum_{k=1}^{n_{N-2}}\left\{\frac{c_{jk}^{(2)}}{c_{jk}^{(1)}}-\frac{c_{kj}^{(2)}}{c_{kj}^{(1)}}\right\},\\ b^{(3)} &= b^{(1)}\sum_{1\leqslant k<l\leqslant n_{N-2}}\left\{\frac{c_{jk}^{(2)}}{c_{jk}^{(1)}}-\frac{c_{kj}^{(2)}}{c_{kj}^{(1)}}\right\}\left\{\frac{c_{jl}^{(2)}}{c_{jl}^{(1)}}-\frac{c_{lj}^{(2)}}{c_{lj}^{(1)}}\right\}\\ &\quad + b^{(1)}\sum_{k=1}^{n_{N-2}}\left\{\frac{c_{jk}^{(3)}}{c_{jk}^{(1)}}-\frac{c_{jk}^{(2)}c_{kj}^{(2)}}{c_{jk}^{(1)}c_{kj}^{(1)}}+\frac{(c_{kj}^{(2)})^2}{(c_{kj}^{(1)})^2}-\frac{c_{kj}^{(3)}}{c_{kj}^{(1)}}\right\}, \end{aligned} \tag{B.6}$$

where c_{jk} and b are defined as

$$\begin{aligned} c_{jk}^{(1)} &= qg_k^{(1)}-pg_j^{(1)},\qquad c_{jk}^{(2)} = qg_k^{(2)}-pg_j^{(2)}+pqg_j^{(1)}g_k^{(1)}\\ c_{jk}^{(3)} &= qg_k^{(3)}-pg_j^{(3)}+pq\big(g_j^{(1)}g_k^{(2)}+g_j^{(2)}g_k^{(1)}\big), \end{aligned} \tag{B.7}$$

and, for $N=3$,

$$b^{(1)}=1,\qquad b^{(2)}=Lpg_j^{(1)},\qquad b^{(3)}=Lpg_j^{(2)}+\tfrac{1}{2}L(L-1)\big(pg_j^{(1)}\big)^2, \tag{B.8}$$

and, for $N>3$,

$$\begin{aligned} b^{(1)} &= 1,\qquad b^{(2)} = \sum_{k=1}^{n_{N-3}}\frac{1-q\lambda_k^{(N-3)}}{1-p\lambda_k^{(N-3)}}pg_j^{(1)},\\ b^{(3)} &= \sum_{1\leqslant k<l\leqslant n_{N-3}}\frac{1-q\lambda_k^{(N-3)}}{1-p\lambda_k^{(N-3)}}\frac{1-q\lambda_l^{(N-3)}}{1-p\lambda_l^{(N-3)}}(pg_j^{(1)})^2\\ &\quad+\sum_{k=1}^{n_{N-3}}\left\{\frac{1-q\lambda_k^{(N-3)}}{1-p\lambda_k^{(N-3)}}pg_j^{(2)}-\frac{\big(1-q\lambda_k^{(N-3)}\big)q\lambda_k^{(N-3)}}{\big(1-p\lambda_k^{(N-3)}\big)^2}\big(pg_j^{(1)}\big)^2\right\}. \end{aligned} \tag{B.9}$$

By solving these equations, the coefficients $g_j^{(k)}$ are uniquely determined. For instance, $g_j^{(1)}$ is simply given by roots of unity:

$$g_j^{(1)} = \exp\left\{\frac{2\pi \mathrm{i}}{n_{N-2}}(j-1)\right\} \quad (1 \leqslant j \leqslant n_{N-2}). \tag{B.10}$$

Substituting (B.5) together with the explicit form of the coefficients $g_j^{(k)}$ into (B.3) and then taking the limit $\lambda_1^{(N-2)} \to 1/p$, one obtains the eigenstate $|\{\lambda^{(N-2)}\}\rangle$ whose elements give $F^{(N-4)\alpha_1\cdots\alpha_{n_{N-3}}}$. Repeating this procedure, one calculates the stationary state.

As a simple example, let us demonstrate this procedure for the maximal sector of $L = N = 3$ ($n_j = L - j$ ($1 \leqslant j \leqslant 2$)). From (B.2) $|\lambda^{(2)}\rangle$ is given by

$$|\lambda^{(2)}\rangle = -p\frac{1-q\lambda_2^{(1)}}{1-p\lambda_2^{(1)}}|2\rangle_1 \otimes |3\rangle_2 - p\frac{1-q\lambda_1^{(1)}}{1-p\lambda_1^{(1)}}|3\rangle_1 \otimes |2\rangle_2. \tag{B.11}$$

Namely F^{23} and F^{32} are, respectively, given by

$$F^{23} = -p\frac{1-q\lambda_2^{(1)}}{1-p\lambda_2^{(1)}}, \qquad F^{32} = -p\frac{1-q\lambda_1^{(1)}}{1-p\lambda_1^{(1)}}. \tag{B.12}$$

To take the limit in (B.3), we expand $\lambda_2^{(1)}$ by solving (B.6). The resultant expression reads

$$\lambda_2^{(1)} = \frac{1}{p} - \left(\lambda_1^{(1)} - 1/p\right) + \frac{p(3p+q)}{p-q}\left(\lambda_1^{(1)} - 1/p\right)^2 + O\left(\left(\lambda_1^{(1)} - 1/p\right)^3\right). \tag{B.13}$$

Inserting this and (B.12) into (B.2), and taking the limit $\lambda_1^{(1)} \to 1/p$, we finally arrive at

$$\begin{aligned} &|\{\lambda^{(1)}\}\rangle = -p(p+q)C^{\alpha_1\alpha_2\alpha_3}|\alpha_1\rangle_1 \otimes |\alpha_2\rangle_2 \otimes |\alpha_3\rangle_3, \\ &C^{123} = C^{231} = C^{312} = 2p+q, \qquad C^{132} = C^{213} = C^{321} = p+2q. \end{aligned} \tag{B.14}$$

We leave it as a future task to extend the concrete calculation as above to the general case and derive an explicit formula for the stationary state. For an alternative approach by the matrix product ansatz, see [PEM].

Appendix C. Eigenvalues and Bethe roots for $N = L = 4$

We list the spectrum of the Hamiltonian (2.6) and the transfer matrix (5.5) in the basic sectors for $N = L = 4$ and $(p, q) = (2/3, 1/3)$. The corresponding Bethe roots with the standard nesting order are also listed here. The spectrum of the Hamiltonian is also obtained by specializing the result in figure 5. The second and the third examples from the last demonstrate that there are two Bethe roots that yield the same eigen-polynomial. These Bethe roots are not regular and the eigen-polynomial is not genuine in the sense of section 5.1.2. The sectors are specified by $(m_1, \ldots, m_4)$ according to (2.14).

Table C1. Table of eigenvalues of the transfer matrix and the Hamiltonian and the corresponding Bethe roots I.

Sector	$\Lambda(\lambda)$	E	$\{\lambda^{(1)}\}$	$\{\lambda^{(2)}\}$	$\{\lambda^{(3)}\}$
(4,0,0,0)	$\frac{16\lambda^4}{27}+1$	0	$\emptyset$	$\emptyset$	$\emptyset$
(1, 3, 0, 0)	$\frac{34\lambda^4}{81}+1$	0	$\{1.5, 1.5, 1.5\}$	$\emptyset$	$\emptyset$
	$\frac{10\lambda^4}{27}+\frac{2\lambda^3}{9}-\frac{2\lambda^2}{3}+2\lambda-1$	-2	$\left\{\begin{matrix}-2.51978\\1.00989-0.565265\mathrm{i}\\1.00989+0.565265\mathrm{i}\end{matrix}\right\}$	$\emptyset$	$\emptyset$
	$\left(\frac{32}{81}\pm\frac{2\mathrm{i}}{81}\right)\lambda^4-\left(\frac{1}{27}\pm\frac{\mathrm{i}}{9}\right)\lambda^3+\left(\frac{1}{3}\mp\frac{\mathrm{i}}{9}\right)\lambda^2+\left(\frac{1}{3}\pm\mathrm{i}\right)\lambda\mp\mathrm{i}$	$-1\pm\frac{\mathrm{i}}{3}$	$\left\{\begin{matrix}-0.809148\pm2.91994\mathrm{i}\\0.824307\pm0.182529\mathrm{i}\\1.16517\mp0.618862\mathrm{i}\end{matrix}\right\}$	$\emptyset$	$\emptyset$
(2, 2, 0, 0)	$\frac{4\lambda^4}{9}+1$	0	$\{1.5, 1.5\}$	$\emptyset$	$\emptyset$
	$\left(\frac{32}{81}\pm\frac{4\mathrm{i}}{81}\right)\lambda^4\mp\frac{2\mathrm{i}\lambda^3}{9}+\frac{5\lambda^2}{9}\pm\mathrm{i}\lambda\mp\mathrm{i}$	$(-1)^2$	$\left\{\begin{matrix}-0.241158\pm1.94173\mathrm{i}\\1.14116\mp0.141729\mathrm{i}\end{matrix}\right\}$	$\emptyset$	$\emptyset$
	$\frac{4\lambda^4}{9}-\frac{2\lambda^3}{3}+3\lambda^2-3\lambda+1$	-3	$\left\{\begin{matrix}-0.75-1.29904\mathrm{i}\\-0.75+1.29904\mathrm{i}\end{matrix}\right\}$	$\emptyset$	$\emptyset$
	$\frac{28\lambda^4}{81}+\frac{8\lambda^3}{27}-\frac{2\lambda^2}{9}+\frac{4\lambda}{3}-1$	$-\frac{4}{3}$	$\left\{\begin{matrix}-3.62132\\0.62132\end{matrix}\right\}$	$\emptyset$	$\emptyset$
	$\frac{28\lambda^4}{81}+\frac{10\lambda^3}{27}-\frac{5\lambda^2}{9}+\frac{5\lambda}{3}-1$	$-\frac{5}{3}$	$\left\{\begin{matrix}-2.42705\\0.927051\end{matrix}\right\}$	$\emptyset$	$\emptyset$
(3, 1, 0, 0)	$\frac{40\lambda^4}{81}+1$	0	$\{1.5\}$	$\emptyset$	$\emptyset$
	$\frac{8\lambda^4}{27}+\frac{8\lambda^3}{9}-\frac{4\lambda^2}{3}+2\lambda-1$	-2	$\{-1.5\}$	$\emptyset$	$\emptyset$
	$\left(\frac{32}{81}\mp\frac{8\mathrm{i}}{81}\right)\lambda^4+\left(\frac{4}{27}\pm\frac{4\mathrm{i}}{9}\right)\lambda^3+\left(\frac{2}{3}\mp\frac{2\mathrm{i}}{9}\right)\lambda^2-\left(\frac{1}{3}\pm\mathrm{i}\right)\lambda\pm\mathrm{i}$	$-1\pm\frac{\mathrm{i}}{3}$	$\{\mp1.5\mathrm{i}\}$	$\emptyset$	$\emptyset$
(1, 1, 2, 0)	$\frac{22\lambda^4}{81}+1$	0	$\{1.5, 1.5, 1.5\}$	$\{1.5, 1.5\}$	$\emptyset$
	$\frac{2\lambda^4}{9}+\frac{2\lambda^3}{9}-\frac{2\lambda^2}{3}+2\lambda-1$	-2	$\left\{\begin{matrix}1.00989-0.565265\mathrm{i}\\1.00989+0.565265\mathrm{i}\\-2.51978\end{matrix}\right\}$	$\{1.5, 1.5\}$	$\emptyset$
	$\left(\frac{20}{81}\pm\frac{2\mathrm{i}}{81}\right)\lambda^4-\left(\frac{1}{27}\pm\frac{\mathrm{i}}{9}\right)\lambda^3+\left(\frac{1}{3}\mp\frac{\mathrm{i}}{9}\right)\lambda^2+\left(\frac{1}{3}\pm\mathrm{i}\right)\lambda\mp\mathrm{i}$	$-1\pm\frac{\mathrm{i}}{3}$	$\left\{\begin{matrix}-0.809148\pm2.91994\mathrm{i}\\0.824307\pm0.182529\mathrm{i}\\1.16517\mp0.618862\mathrm{i}\end{matrix}\right\}$	$\{1.5, 1.5\}$	$\emptyset$
	$\left(\frac{2}{9}\pm\frac{4\mathrm{i}}{81}\right)\lambda^4\mp\frac{2\mathrm{i}\lambda^3}{9}+\frac{5\lambda^2}{9}\pm\mathrm{i}\lambda\mp\mathrm{i}$	$(-1)^2$	$\left\{\begin{matrix}-0.241158\pm1.94173\mathrm{i}\\1.14116\mp0.141729\mathrm{i}\\1.5\end{matrix}\right\}$	$\left\{\begin{matrix}0.69207\pm0.648316\mathrm{i}\\1.19884\mp0.161286\mathrm{i}\end{matrix}\right\}$	$\emptyset$
	$\frac{22\lambda^4}{81}-\frac{2\lambda^3}{3}+3\lambda^2-3\lambda+1$	-3	$\left\{\begin{matrix}1.5, -0.75-1.29904\mathrm{i}\\-0.75+1.29904\mathrm{i}\end{matrix}\right\}$	$\left\{\begin{matrix}0.789474-0.957186\mathrm{i}\\0.789474+0.957186\mathrm{i}\end{matrix}\right\}$	$\emptyset$
	$\frac{14\lambda^4}{81}+\frac{8\lambda^3}{27}-\frac{2\lambda^2}{9}+\frac{4\lambda}{3}-1$	$-\frac{4}{3}$	$\left\{\begin{matrix}1.5, -3.62132\\0.62132\end{matrix}\right\}$	$\left\{\begin{matrix}0.686441-0.503364\mathrm{i}\\0.686441+0.503364\mathrm{i}\end{matrix}\right\}$	$\emptyset$
	$\frac{14\lambda^4}{81}+\frac{10\lambda^3}{27}-\frac{5\lambda^2}{9}+\frac{5\lambda}{3}-1$	$-\frac{5}{3}$	$\left\{\begin{matrix}1.5, -2.42705\\0.927051\end{matrix}\right\}$	$\left\{\begin{matrix}0.765957-0.188811\mathrm{i}\\0.765957+0.188811\mathrm{i}\end{matrix}\right\}$	$\emptyset$
	$\frac{10\lambda^4}{81}+\frac{2\lambda^3}{9}+\frac{4\lambda^2}{3}-2\lambda+1$	-2	$\left\{\begin{matrix}-0.61352-2.06536\mathrm{i}\\-0.61352+2.06536\mathrm{i}\\0.72704\end{matrix}\right\}$	$\left\{\begin{matrix}-9.97723\\0.977226\end{matrix}\right\}$	$\emptyset$
	$\left(\frac{8}{81}\pm\frac{10\mathrm{i}}{81}\right)\lambda^4+\left(\frac{13}{27}\mp\mathrm{i}\right)\lambda^3+\left(\frac{1}{3}\pm\frac{25\mathrm{i}}{9}\right)\lambda^2-\left(\frac{1}{3}\pm3\mathrm{i}\right)\lambda\pm\mathrm{i}$	$-3\pm\frac{\mathrm{i}}{3}$	$\left\{\begin{matrix}-1.97293\mp0.68627\mathrm{i}\\-0.405293\pm1.74466\mathrm{i}\\0.762834\mp0.481463\mathrm{i}\end{matrix}\right\}$	$\left\{\begin{matrix}-3.45362\pm7.73728\mathrm{i}\\1.07431\mp0.289005\mathrm{i}\end{matrix}\right\}$	$\emptyset$
(1, 2, 1, 0)	$\frac{26\lambda^4}{81}+1$	0	$\{1.5, 1.5, 1.5\}$	$\{1.5\}$	$\emptyset$
	$\frac{22\lambda^4}{81}+\frac{2\lambda^3}{9}-\frac{2\lambda^2}{3}+2\lambda-1$	-2	$\left\{\begin{matrix}1.00989-0.565265\mathrm{i}\\1.00989+0.565265\mathrm{i}\\-2.51978\end{matrix}\right\}$	$\{1.5\}$	$\emptyset$
	$\left(\frac{8}{27}\pm\frac{2\mathrm{i}}{81}\right)\lambda^4-\left(\frac{1}{27}\pm\frac{\mathrm{i}}{9}\right)\lambda^3+\left(\frac{1}{3}\mp\frac{\mathrm{i}}{9}\right)\lambda^2+\left(\frac{1}{3}\pm\mathrm{i}\right)\lambda\mp\mathrm{i}$	$-1\pm\frac{\mathrm{i}}{3}$	$\left\{\begin{matrix}-0.809148\pm2.91994\mathrm{i}\\0.824307\pm0.182529\mathrm{i}\\1.16517\mp0.618862\mathrm{i}\end{matrix}\right\}$	$\{1.5\}$	$\emptyset$
	$\frac{10\lambda^4}{81}+\frac{8\lambda^3}{9}-\frac{4\lambda^2}{3}+2\lambda-1$	-2	$\{-1.5, 1.5, 1.5\}$	$\{-0.5\}$	$\emptyset$
	$\left(\frac{2}{9}\mp\frac{8\mathrm{i}}{81}\right)\lambda^4+\left(\frac{4}{27}\pm\frac{4\mathrm{i}}{9}\right)\lambda^3+\left(\frac{2}{3}\mp\frac{2\mathrm{i}}{9}\right)\lambda^2-\left(\frac{1}{3}\pm\mathrm{i}\right)\lambda\pm\mathrm{i}$	$-1\pm\frac{\mathrm{i}}{3}$	$\{\mp1.5\mathrm{i}, 1.5, 1.5\}$	$\{0.253846\mp0.969231\mathrm{i}\}$	$\emptyset$

Table C2. Table of eigenvalues of the transfer matrix and the Hamiltonian and the corresponding Bethe roots II.

Sector	$\Lambda(\lambda)$	E	$\{\lambda^{(1)}\}$	$\{\lambda^{(2)}\}$	$\{\lambda^{(3)}\}$
(1,2,1,0)	$\frac{16\lambda^4}{81}+\frac{\lambda^3}{3}+\lambda-1$	-1	$\{-7.5494,\ 0.383413,\ 1.16599\}$	$\{1.5\}$	$\emptyset$
	$\left(\frac{2}{81}\mp\frac{14i}{81}\right)\lambda^4+\left(\frac{10}{9}\pm\frac{11i}{9}\right)\lambda^3-\left(\frac{5}{9}\pm 3i\right)\lambda^2\pm 3i\lambda\mp i$	$(-3)^2$	$\{-1.9214\pm 0.238512i,\ -0.315515\mp 1.62413i,\ 1.00162\pm 0.326798i\}$	$\{-2.19231\mp 3.46154i\}$	$\emptyset$
	$\frac{4\lambda^4}{27}+\frac{7\lambda^3}{27}+\frac{14\lambda^2}{9}-\frac{7\lambda}{3}+1$	$-\frac{7}{3}$	$\{-0.511746-1.74231i,\ -0.511746+1.74231i,\ 1.02349\}$	$\{1.5\}$	$\emptyset$
	$\frac{2\lambda^4}{9}-\frac{4\lambda^3}{27}+\frac{20\lambda^2}{9}-\frac{8\lambda}{3}+1$	$-\frac{8}{3}$	$\{-0.717003-1.5691i,\ -0.717003+1.5691i,\ 1.13401\}$	$\{-0.214286\}$	$\emptyset$
(2, 1, 1, 0)	$\frac{28\lambda^4}{81}+1$	0	$\{1.5, 1.5\}$	$\{1.5\}$	$\emptyset$
	$\left(\frac{8}{27}\pm\frac{4i}{81}\right)\lambda^4\mp\frac{2i\lambda^3}{9}+\frac{5\lambda^2}{9}\pm i\lambda\mp i$	$(-1)^2$	$\{-0.241158\pm 1.94173i,\ 1.14116\mp 0.141729i\}$	$\{1.5\}$	$\emptyset$
	$\frac{28\lambda^4}{81}-\frac{2\lambda^3}{3}+3\lambda^2-3\lambda+1$	-3	$\{-0.75-1.29904i,\ -0.75+1.29904i\}$	$\{1.5\}$	$\emptyset$
	$\frac{20\lambda^4}{81}+\frac{8\lambda^3}{27}-\frac{2\lambda^2}{9}+\frac{4\lambda}{3}-1$	$-\frac{4}{3}$	$\{-3.62132,\ 0.62132\}$	$\{1.5\}$	$\emptyset$
	$\frac{20\lambda^4}{81}+\frac{10\lambda^3}{27}-\frac{5\lambda^2}{9}+\frac{5\lambda}{3}-1$	$-\frac{5}{3}$	$\{-2.42705,\ 0.927051\}$	$\{1.5\}$	$\emptyset$
	$\frac{4\lambda^4}{27}+\frac{8\lambda^3}{9}-\frac{4\lambda^2}{3}+2\lambda-1$	-2	$\{-1.5, 1.5\}$	$\{0.3\}$	$\emptyset$
	$\left(\frac{20}{81}\mp\frac{8i}{81}\right)\lambda^4+\left(\frac{4}{27}\pm\frac{4i}{9}\right)\lambda^3+\left(\frac{2}{3}\mp\frac{2i}{9}\right)\lambda^2-\left(\frac{1}{3}\pm i\right)\lambda\pm i$	$-1\pm\frac{i}{3}$	$\{\mp 1.5i, 1.5\}$	$\{0.617647\mp 0.529412i\}$	$\emptyset$
	$\frac{4\lambda^4}{81}+\frac{8\lambda^3}{9}+\frac{2\lambda^2}{3}-2\lambda+1$	-2	$\{-1.5i, 1.5i\}$	$\{-4.5\}$	$\emptyset$
	$\left(-\frac{4}{81}\mp\frac{16i}{81}\right)\lambda^4+\left(\frac{44}{27}\pm\frac{4i}{3}\right)\lambda^3-\left(\frac{4}{3}\pm\frac{28i}{9}\right)\lambda^2+\left(\frac{1}{3}\pm 3i\right)\lambda\mp i$	$-3\pm\frac{i}{3}$	$\{-1.5, \mp 1.5i\}$	$\{-2.55882\mp 4.76471i\}$	$\emptyset$
(1,1,1,1)	$\frac{14\lambda^4}{81}+1$	0	$\{1.5, 1.5, 1.5\}$	$\{1.5, 1.5\}$	$\{1.5\}$
	$\frac{10\lambda^4}{81}+\frac{2\lambda^3}{9}-\frac{2\lambda^2}{3}+2\lambda-1$	-2	$\{1.00989-0.565265i,\ 1.00989+0.565265i,\ -2.51978\}$	$\{1.5, 1.5\}$	$\{1.5\}$
	$\left(\frac{4}{27}\pm\frac{2i}{81}\right)\lambda^4-\left(\frac{1}{27}\pm\frac{i}{9}\right)\lambda^3+\left(\frac{1}{3}\mp\frac{i}{9}\right)\lambda^2+\left(\frac{1}{3}\pm i\right)\lambda\mp i$	$-1\pm\frac{i}{3}$	$\{-0.809148\pm 2.91994i,\ 0.824307\pm 0.182529i,\ 1.16517\mp 0.618862i\}$	$\{1.5, 1.5\}$	$\{1.5\}$
	$\left(\frac{10}{81}\pm\frac{4i}{81}\right)\lambda^4\mp\frac{2i\lambda^3}{9}+\frac{5\lambda^2}{9}\pm i\lambda\mp i$	$(-1)^2$	$\{-0.241158\pm 1.94173i,\ 1.14116\mp 0.141729i,\ 1.5\}$	$\{0.69207\pm 0.648316i,\ 1.19884\mp 0.161286i\}$	$\{1.5\}$
	$\frac{14\lambda^4}{81}-\frac{2\lambda^3}{3}+3\lambda^2-3\lambda+1$	-3	$\{-0.75-1.29904i,\ -0.75+1.29904i,\ 1.5\}$	$\{0.789474-0.957186i,\ 0.789474+0.957186i\}$	$\{1.5\}$
	$\frac{2\lambda^4}{27}+\frac{8\lambda^3}{27}-\frac{2\lambda^2}{9}+\frac{4\lambda}{3}-1$	$-\frac{4}{3}$	$\{1.5, -3.62132,\ 0.62132\}$	$\{0.686441-0.503364i,\ 0.686441+0.503364i\}$	$\{1.5\}$
	$\frac{2\lambda^4}{27}+\frac{10\lambda^3}{27}-\frac{5\lambda^2}{9}+\frac{5\lambda}{3}-1$	$-\frac{5}{3}$	$\{1.5, -2.42705,\ 0.927051\}$	$\{0.765957-0.188811i,\ 0.765957+0.188811i\}$	$\{1.5\}$
	$\frac{2\lambda^4}{81}+\frac{2\lambda^3}{9}+\frac{4\lambda^2}{3}-2\lambda+1$	-2	$\{-0.61352-2.06536i,\ -0.61352+2.06536i,\ 0.72704\}$	$\{-9.97723,\ 0.977226\}$	$\{1.5\}$
	$\pm\frac{10i}{81}\lambda^4+\left(\frac{13}{27}\mp i\right)\lambda^3+\left(\frac{1}{3}\pm\frac{25i}{9}\right)\lambda^2-\left(\frac{1}{3}\pm 3i\right)\lambda\pm i$	$-3\pm\frac{i}{3}$	$\{-1.97293\mp 0.68627i,\ -0.405293\pm 1.74466i,\ 0.762834\mp 0.481463i\}$	$\{-3.45362\pm 7.73728i,\ 1.07431\mp 0.289005i\}$	$\{1.5\}$
	$\frac{4\lambda^4}{81}+\frac{\lambda^3}{3}+\lambda-1$	-1	$\{1.16599, -7.5494,\ 0.383413\}$	$\{0.5, 1.5\}$	$\{0.954545\}$
	$\left(-\frac{10}{81}\mp\frac{14i}{81}\right)\lambda^4+\left(\frac{10}{9}\pm\frac{11i}{9}\right)\lambda^3-\left(\frac{5}{9}\pm 3i\right)\lambda^2\pm 3i\lambda\mp i$	$(-3)^2$	$\{-1.9214\pm 0.238512i,\ -0.315515\mp 1.62413i,\ 1.00162\pm 0.326798i\}$	$\{-2.19231\mp 3.46154i,\ 1.5\}$	$\{-0.101751\mp 0.59081i\}$

Table C3. Table of eigenvalues of the transfer matrix and the Hamiltonian and the corresponding Bethe roots III.

Sector	$\Lambda(\lambda)$	E	$\{\lambda^{(1)}\}$	$\{\lambda^{(2)}\}$	$\{\lambda^{(3)}\}$
(1,1,1,1)					
	$\frac{7\lambda^3}{27}+\frac{14\lambda^2}{9}-\frac{7\lambda}{3}+1$	$-\frac{7}{3}$	$\left\{\begin{matrix}-0.511746-1.74231i\\-0.511746+1.74231i\\1.02349\end{matrix}\right\}$	$\{-1.5, 1.5\}$	$\{0.3\}$
	$\frac{2\lambda^4}{27}-\frac{4\lambda^3}{27}+\frac{20\lambda^2}{9}-\frac{8\lambda}{3}+1$	$-\frac{8}{3}$	$\left\{\begin{matrix}-0.717003-1.5691i\\-0.717003+1.5691i\\1.13401\end{matrix}\right\}$	$\left\{\begin{matrix}-0.214286\\1.5\end{matrix}\right\}$	$\{0.672414\}$
	$-\frac{10\lambda^4}{81}+\frac{8\lambda^3}{9}+\frac{2\lambda^2}{3}-2\lambda+1$	-2	$\{-1.5i, 1.5i, 1.5\}$	$\{-2.06427, 0.778553\}$	$\{-1.77273\}$
	$\left(-\frac{2}{9}\mp\frac{16i}{81}\right)\lambda^4+\left(\frac{44}{27}\pm\frac{4i}{3}\right)\lambda^3 -\left(\frac{4}{3}\pm\frac{28i}{9}\right)\lambda^2+\left(\frac{1}{3}\pm 3i\right)\lambda\mp i$	$-3\pm\frac{i}{3}$	$\{-1.5, \mp 1.5i, 1.5\}$	$\left\{\begin{matrix}-2.35155\mp 3.02941i\\0.874627\mp 0.15521i\end{matrix}\right\}$	$\{-1.63706\mp 1.91878i\}$
	$-\frac{2\lambda^4}{81}+\frac{8\lambda^3}{9}-\frac{4\lambda^2}{3}+2\lambda-1$	-2	$\{-1.5, 1.5, 1.5\}$	$\{-0.5, 1.5\}$	$\{0.576923\}$
				$\{1.5, 1.5\}$	$\{-1.5\}$
	$\left(\frac{2}{27}\mp\frac{8i}{81}\right)\lambda^4+\left(\frac{4}{27}\pm\frac{4i}{9}\right)\lambda^3 +\left(\frac{2}{3}\mp\frac{2i}{9}\right)\lambda^2-\left(\frac{1}{3}\pm i\right)\lambda\pm i$	$-1\pm\frac{i}{3}$	$\{\mp 1.5i, 1.5, 1.5\}$	$\left\{\begin{matrix}0.253846\mp 0.969231i\\1.5\end{matrix}\right\}$	$\{0.784404\mp 0.385321i\}$
				$\{1.5, 1.5\}$	$\{\mp 1.5i\}$
	$-\frac{50\lambda^4}{81}+4\lambda^3-6\lambda^2+4\lambda-1$	-4	$\{-1.5, -1.5i, 1.5i\}$	$\left\{\begin{matrix}-0.5-3.74166i\\-0.5+3.74166i\end{matrix}\right\}$	$\{25.5\}$

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